Two proofs on rational tangles

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Very short proofs are given for two basic facts about rational tangles.

Definition (1) A rational tangle is an embedding of two arcs in a 3–ball $Q$ ambient homeomorphic to a trivial pair of arcs (the trivial tangle).

Definition (2) (See Figure 1) Take a 3-braid $\beta$ embedded in a ball $B^2 \times I$ and at the top join arcs 2 and 3 by a hook. Extend the ball to $Q = B^2 \times J$ (to enclose the hook) where $J = [-1, 1]$ and extend the remaining arcs to the boundary of $Q$. This is the same as the “twist” definition of a rational tangle where the remaining top string is NW. More precisely, let $s$ and $t$ be the two standard generators of the 3–braid group (ie $\sigma_1$ and $\sigma_2$) then any braid is a word of the form $s^{a_1}t^{b_1}\ldots$ and this corresponds to $a_1$ (RH) twists of strings 1 and 2 (at the bottom) followed by $b_1$ of strings 2 and 3 etc.

Call this the tangle presented by $\beta$. Note that you get exactly the same set of tangles (up to homeo) if you attach the hook to strings 1 and 2 instead of 2 and 3. Call the tangle obtained by placing the hook on strings 1 and 2 of $\beta$, the alternative tangle presented by $\beta$.

Proposition The two definitions conicide.
Proof \((2) \implies (1)\) Each twist extends to a homeo of \(Q\).

\((1) \implies (2)\) Make the trivial tangle by using the trivial braid. Call the given homeo \(h\).

By standard methods (disc theorem and collaring) we can assume that \(h\) is the identity near the top (in \(B^3 \times [-1, -1/2]\) say) and on the sides (\(\partial B^2 \times J\)). Pull the hook in the trivial tangle up (following the arc \(\alpha\) in Figure 1) by an ambient isotopy until it is inside the identity region and pull this motion back (by \(h^{-1}\)) to an isotopy of \(h\). In other words we pull a hook on one of the strings of the given tangle up into the identity region using the arc \(h^{-1}\alpha\).

We now have \(h\) the identity except in \(Q' = B^2 \times [-1/2, 1]\) where it maps a 3-string tangle on the left (the original tangle with a hook removed from one string) to the trivial 3-braid. Moreover \(h\) is the identity on the top and sides of \(Q'\). The restriction of \(h\) to the bottom (and the identity on the rest of the boundary) extends by the cone construction (cone point \((0, 0, 0)\)) to a level-preserving homeo \(h'\) of \(Q'\) to itself. \(h'\) pulls the trivial braid in the codomain back to a braid \(\beta\) say in the domain. \(g = (h')^{-1} \circ h\) is a homeo of our original tangle to the tangle presented by \(\beta\). Furthermore by the Alexander trick \(g\) is ambient isotopic rel boundary to the identity map.

Proposition Any rational tangle has a braid presentation (possible in alternative form) using a braid of the form \(s^a t^b \ldots\) with EITHER all \(a\)’s positive and all \(b\)’s negative OR all \(a\)’s negative and all \(b\)’s positive.

Call such a presentation standard form. Standard form is precisely the same as braid presentation using an alternating braid.

Corollary Rational tangles (and hence rational knots) are alternating.

Proof If the braid word is reduced (no \(ss^{-1}\)’s etc) and not alternating it must contain one of the following subwords:

\[st \quad ts \quad s^{-1}t^{-1} \quad t^{-1}s^{-1}\]

In each case we can isotope the tangle to get a new braid presentation of shorter braid length. After a finite number of such isotopies the braid is in standard form. We explain how to deal with \(st\); the other cases are analogous. Refer to Figure 2 where \(\beta = \phi_{st}\psi\).

Let \(A\) be the level just below \(st\) and turn the piece above \(A\) a half-turn then the left. You can think of this as a flype if you like, except that there are THREE strings at the bottom of the flype box and only ONE at the top. After this isotopy the two original crossings above \(A\) have been replaced by one new crossing.

Notice that the proof is algorithmic and that you end with an alternative presentation iff there were an odd number of shortening moves.
Remark This proof is adapted from the first (easy part) of Berger’s algorithm to minimise the length of a 3-braid word [J Phy A 27 (1994) 6205–6213].