# The compression theorem III: applications

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**Abstract**

This is the third of three papers about the *Compression Theorem*: if $M^m$ is embedded in $Q^q \times \mathbb{R}$ with a normal vector field and if $q - m \geq 1$, then the given vector field can be *straightened* (ie, made parallel to the given $\mathbb{R}$ direction) by an isotopy of $M$ and normal field in $Q \times \mathbb{R}$.

The theorem can be deduced from Gromov's theorem on directed embeddings [4; 2.4.5 (C′)] and the first two parts gave proofs. Here we are concerned with applications.

We give short new (and constructive) proofs for immersion theory and for the loops–suspension theorem of James et al and a new approach to classifying embeddings of manifolds in codimension one or more, which leads to theoretical solutions.

We also consider the general problem of controlling the singularities of a smooth projection up to $C^0$–small isotopy and give a theoretical solution in the codimension $\geq 1$ case.

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## 1 Introduction

We work throughout in the smooth ($C^\infty$) category. Embeddings, immersions, regular homotopies etc will be assumed either to take boundary to boundary or to meet the boundary in a codimension 0 submanifold of the boundary. Thus for example if $f\colon M \to Q$ is an immersion then we assume that either $f^{-1}\partial Q = \partial M$ or $f^{-1}\partial Q$ is a codimension 0 submanifold of $\partial M$. In the latter case we speak of $M$ having *relative boundary*. The tangent bundle of a manifold $W$ is denoted $T(W)$ and the tangent space at $x \in W$ is denoted $T_x(W)$. Throughout the paper, "normal" means independent (as in the usual meaning of "normal bundle") and not necessarily perpendicular.

This is the third of a set of three papers about the following result:

**Compression Theorem** *Suppose that $M^m$ is embedded in $Q^q \times \mathbb{R}$ with a normal vector field and suppose that $q - m \geq 1$. Then the vector field can be straightened (ie, made parallel to the given $\mathbb{R}$ direction) by an isotopy of $M$ and normal field in $Q \times \mathbb{R}$.*

Thus the theorem moves $M$ to a position where it projects by vertical projection (ie "compresses") to an immersion in $Q$.

The theorem can be deduced from Gromov's theorem on directed embeddings [4; 2.4.5 (C′)]. Proofs are given in parts I and II [14, 15].

Immersion theory [5, 17] implies the embedding is *regularly homotopic* to an immersion which covers an immersion in $Q$ and using configuration space models of multiple-loops-suspension spaces [6, 11, 12, 16] it can be seen that the embedding is *bordant* (by a bordism mapping to $M$) to an embedding which covers an immersion in $Q$, see [9]. Thus the new information which the compression theorem provides is that the embedding is *isotopic* to an embedding covering an immersion in $Q$. Moreover we can apply the compression theorem to give short and constructive proofs for both immersion theory and configuration space theory.

The Compression Theorem also sheds light on the following problem:

$C^0$**–Singularity Problem** *Given $M \subset W$ and $p\colon W \to Q$ a submersion, how much control do we have over the singularities of $p|M$ if we are allowed a $C^0$–small isotopy of $M$ in $W$?*

The problem includes the problem of controlling the singularities of a map $f\colon M \to Q$ by a $C^0$–small homotopy. This is because we can always factor $f$ as $f \times q\colon M \subset Q \times \mathbb{R}^t \xrightarrow{\text{proj}} Q$ where $q\colon M \to \mathbb{R}^t$ is an embedding.

The Compression Theorem gives a necessary and sufficient condition for desingularising the projection in the case that $\dim(W) = \dim(Q)+1$ and $\dim(M) < \dim(Q)$, namely that there should exist an appropriate normal line field, and it can be extended to give necessary and sufficient conditions for singularities of almost any pre-specified type, namely that there should exist a line field with those singularities. Both these results extend to the case with just the hypothesis $\dim(M) < \dim(Q)$ where "line field" is replaced by "plane field". They also have natural relative and parametrised versions.

It is worth contrasting these results with the $C^\infty$–singularity problem ("$C^0$–small" is replaced by "$C^\infty$–small") where there is the classical Thom–Boardman classification of $C^\infty$–stable singularities [1]. For most topological purposes (eg for applications to homotopy theory) the $C^0$ classification is more natural than the classical $C^\infty$ classification. Furthermore essential singularities (up to $C^0$ homotopy) have natural interpretations as generalised bordism characteristic classes similar to those investigated by Korschorke [7] (see section 5).

Our methods also give information in the negative codimension case (ie where $\dim(M) \geq \dim(Q)$) which will be investigated in a future paper. See also Spring [18].

Two examples of immediate application of the compression theorem are the following:

**Corollary 1.1** *Let $\pi$ be a group. There is a classifying space $BC(\pi)$ such that the set of homotopy classes $[Q, BC(\pi)]$ is in natural bijection with the set of cobordism classes of framed submanifolds $L$ of $Q \times \mathbb{R}$ of codimension 2 equipped with a homomorphism $\pi_1(Q \times \mathbb{R} - L) \to \pi$.*

**Corollary 1.2** *Let $Q$ be a connected manifold with basepoint $*$ and let $M$ be any collection of disjoint submanifolds of $Q - \{*\}$ each of which has codimension $\geq 2$ and is equipped with a normal vector field. Define the* vertical *loop space of $Q$ denoted $\Omega^{\mathrm{vert}}(Q)$ to comprise loops which meet given tubular neighbourhoods of manifolds in $M$ in straight line segments parallel to the given vector field. Then the natural inclusion $\Omega^{\mathrm{vert}}(Q) \subset \Omega(Q)$ (where $\Omega(Q)$ is the usual loop space) is a weak homotopy equivalence.*

**Proofs** The first corollary is a special case of the classification theorem for links in codimension 2 given in [3; theorem 4.15], the space $BC(\pi)$ being the rack space of the conjugacy rack of $\pi$. To prove the second corollary suppose given a based map $f\colon S^n \to \Omega(Q)$ then the adjoint of $f$ can be regarded as a map $g\colon S^n \times \mathbb{R} \to Q$ which takes the ends of $S^n \times \mathbb{R}$ and $\{*\} \times \mathbb{R}$ to the basepoint. Make $g$ transverse to $M$ to create a number of manifolds embedded in $S^n \times \mathbb{R}$ and equipped with normal vector fields. Apply the compression theorem to each of these (the local version proved in section 4 of this paper). The result is to deform $g$ into the adjoint of a map $S^n \to \Omega^{\mathrm{vert}}(Q)$. This shows that $\Omega^{\mathrm{vert}}(Q) \subset \Omega(Q)$ induces a surjection on $\pi_n$. A similar argument applied to a homotopy, using the relative compression theorem, proves injectivity. (The vertical loop space is introduced in Wiest [20]; for applications and related results see [20, 21].) $\square$

More substantial applications are given in sections 1 (immersion theory), section 2 (loops–suspension theory), section 3 (embeddings and knots) and section 4 (simplifying singuratities).

We would like to thank Bert Wiest for observing corollary 1.2 and Yasha Eliashberg for helpful comments on previous versions of this paper.

## 2 Immersion theory

In the compression theorem, the existence of the immersion of $M$ in $Q$ follows from immersion theory; however immersion theory gives us no explicit information about this immersion, which is only determined up to regular homotopy. By contrast the compression theorem gives us an explicit description of the immersion in terms of the given embedding and normal vector field in $Q \times \mathbb{R}$. Moreover the compression theorem can be used to give a new proof for immersion theory as we now show. We start by proving the simplest statement of immersion theory and then develop the proof into a full statement.

Let $M$ be an $n$–manifold. We shall explicitly describe a way of *rotating the fibres* of the tangent bundle $TM$ into $M$. Regard the zero section $M$ as 'vertical' and the fibres as 'horizontal'. Consider $TM$ as a smooth $2n$–manifold, then its tangent bundle restricted to $M$ is the Whitney sum $TM \oplus TM$. The two copies of $TM$ are the vertical copy parallel to $M$ and the horizontal copy parallel to the fibres of $TM$. Each vector $v \in TM$ then determines two vectors $v_v$ and $v_h$ in $TM \oplus TM$ which span a plane. In this plane we can 'rotate' $v_h$ to $v_v$. Since we are not at this moment considering a particular metric 'rotation' needs to be defined: to be precise we consider the family of linear transformations of this plane given by

**2.1**
$$v_h \mapsto cv_h + sv_v, \quad v_v \mapsto cv_v - sv_h \qquad \text{where}$$
$$c = \cos \tfrac{\pi}{2}t, \quad s = \sin \tfrac{\pi}{2}t, \quad 0 \le t \le 1.$$

This formula (applied to each such plane) determines a bundle isotopy (a 1–parameter family of bundle isomorphisms) which is the required rotation of the fibres of $TM$ into $M$.

**Simple Immersion Theorem 2.2** *Suppose that we are given a bundle monomorphism $f\colon TM \to TQ$ (ie, a map $M \to Q$ covered by a vector space monomorphism on each fibre) and that either $q - m \ge 1$ or $q - m \ge 0$ and each component of $M$ has relative boundary. Then the restriction $f|\colon M \to Q$ is homotopic to an immersion.*

**Proof** Composing $f$ with an exponential map for $TQ$ gives a map $g\colon TM \to Q$ which embeds fibres into $Q$. Choose an embedding $q\colon M \to \mathbb{R}^n$ for some $n$ and also denote by $q$ the map $TM \to \mathbb{R}^n$ given by projecting the bundle $TM$ onto $M$ (the usual bundle projection) and then composing with $q$. We then have the embedding $g \times q\colon TM \to Q \times \mathbb{R}^n$. The fibres of $TM$ are embedded parallel to $Q$ and the $n$ directions parallel to the axes of $\mathbb{R}^n$ determine $n$ independent vector fields at $M$ normal to the fibres of $TM$.

Now choose a complement for $T(TM)\,|\,M \cong TM \oplus TM$ in $T(Q\times\mathbb{R})\,|\,M$. Then the rotation of the fibres of $TM$ into $M$ (formula 2.1 above) extends (by the identity on the complement) to a bundle isotopy of $T(Q\times\mathbb{R})\,|\,M$ which carries the these $n$ fields normal to $M$ to yield $n$ independent normal fields. The result now follows from the multi-compression theorem in part I [14; corollary 4.5]. □

The proof of the simple immersion theorem just given is very explicit, which contrasts with the standard Hirsch–Smale approach [5, 17] or the proofs given by Gromov [4]. Given a particular bundle monomorphism $TM \to TQ$ the proof can be used to construct a homotopic immersion $M \to Q$. The only serious element of choice in the proof is the embedding of $M$ in $\mathbb{R}^n$. It is worth remarking that Eliashberg and Gromov [2; Theorem 4.3.4] have also given a short proof of immersion theory which yields an explicit immersion.

We now extend the proof to give a parametrised version. The usual statement of immersion theory will follow easily. Notice that, in the proof just given, there was an explicit bundle homotopy of the $n$ independent normal fields to $M$ to the vertical fields. Thus using the Normal Deformation Theorem in place of the multi-compression theorem, there is an immediate parametrised version of the proof. However, it is worth repeating the proof to construct the parametrised version because it can be done in a particularly explicit fashion.

Let $F\colon K \times M \to Q$ be a smooth map such that $F\,|\,\{t\} \times M \to Q$ is an immersion for each $t \in K$, where $K$ is a smooth manifold. Call such a map a *$K$–family* of immersions of $M$ in $Q$. There is a similar notion of a *$K$–family* of bundle monomorphisms $G\colon K\times TM \to TQ$. Given an immersion $f$ of $M$ in $Q$ then the derivative $Tf$ of $f$ is a bundle monomorphism. By differentiating a $K$–family $F$ of immersions for each $t \in K$ we obtain a $K$–family of bundle monomorphisms which we denote $TF$.

**Full Immersion Theorem 2.3** *Let $K, M, Q$ be smooth manifolds and suppose $q-m \geq 1$. Suppose $G\colon K\times TM \to TQ$ is a $K$–family of bundle monomorphisms then there is a $K$–family of immersions $F\colon K \times M \to Q$ such that $TF$ is homotopic to $G$ through bundle monomorphisms.*

**Proof** We construct $F$ as in theorem 2.2 but using the parametrised version of the multi-compression theorem. (The parameter space $K$ plays no real rôle in the proof and we shall not mention it again.) The only point that needs to be checked is that the vector fields are grounded as we come to them and can therefore be straightened inductively.

The following considerations explain why the fields start grounded. We choose metrics so that distances in $\mathbb{R}^n$ are very large compared to distances in $Q$.

This means that the embedding of $M$ in $Q \times \mathbb{R}^n$ is roughly parallel to $\mathbb{R}^n$. Number the fields 1 to $n$ corresponding to the $n$ coordinate axes in $\mathbb{R}^n$ and say that field $i$ is grounded if it is never parallel to the negative $i$–axis. Now before the rotation of the fibres of $TM$ into $M$, the fields are all parallel to the corresponding positive axes. To produce a point where field $i$ say is not grounded requires a rotation of that field through $\pi$. But the rotation of the fibres of $TM$ into $M$ is close to a genuine metric rotation through $\pi/2$, which is not far enough to unground one of our vector fields, which are therefore all grounded before we start to compress. So we just have to make sure that each compression maintains the property that subsequent fields are grounded.

To see this we shall modify the method of producing the initial $n$ normal fields so that the fields start independent and normal and with each field nearly straight (ie, field $i$ makes a small angle with the positive $i$–axis for each $i$). We then straighten the fields in turn making sure that the straightening process disturbs the angles of subsequent fields only a little. This means that after each straightening subsequent fields are still nearly straight and the process continues.

So we start by finding the $n$ independent, normal, nearly straight fields. Suppose given a real number $\mu > 0$. Then we can choose the scale factor for distances in $\mathbb{R}^n$ compared to those in $Q$ to be sufficiently large so that the following is true. Consider a point $x \in M$ and consider rotation of the fibres of $TM$ through an 'angle' $\pi/2 - \mu$ (ie, use formula 2.1 with $t = 1 - 2\mu/\pi$). If $\{v_1, \dots, v_m\}$ is a basis for $T_xM$ (in this partially rotated position) and if $\{e_1, \dots, e_n\}$ is standard basis for $T_x\mathbb{R}^n$, then the vectors $\{v_1, \dots, v_m, e_1, \dots, e_n\}$ will form an $(n+m)$–frame at $x$.

Now finish the rotation of the fibres of $TM$ into $M$ (ie, apply the linear transformation given by formula 2.1 with $t = 2\mu/\pi$) and extend to a bundle isomorphism of $T(Q \times \mathbb{R}) \,|\, M$ by using the orthogonal complement of $TM \oplus TM$ in $T(Q \times \mathbb{R}) \,|\, M$. Let this extension carry $\{e_1, \dots, e_n\}$ to $\{f_1, \dots, f_n\}$ say. Then $\{f_1, \dots, f_n\}$ are independent, normal to $M$ at $x$ and nearly parallel to $\{e_1, \dots, e_n\}$. Doing the same manoeuvre at each point of $M$ constructs the required normal fields.

Now comes the only subtle point in the argument. We are inductively straightening nearly straight fields. This can be done by a small isotopy using the global proof given in section 2. However there is no reason to assume that this isotopy will be small *enough.* Note that the multi-compression theorem uses the *local* compression theorem since it is applied inside an induced regular neighbourhood and, as we proceed with our inductive straightening process, these induced neighbourhoods may become truly tiny. So we have to check that the property that fields remain nearly straight is not disturbed by application of the local proof of the compression theorem.

Look carefully at the proof of the Local Compression Theorem [14; 4.4]. It starts with a perpendicular vector field. Suppose that $\rho$ is the given nearly straight field which we intend to straighten. Let $\alpha$ be the corresponding perpendicular field (obtained by othogonalising $\rho$ with respect to $TM$), and let $\beta$ be the result of upward rotation applied to $\alpha$.

Now $\beta$ and $\rho$ both make an angle less than $\pi/2$ with $\alpha$ which is perpendicular to $M$. It follows from elementary spherical geometry that the straight line isotopy which moves $\rho$ to $\beta$ is disjoint from $M$. Moreover by definition of upwards rotation, $\beta$ is at least as close to vertical as $\rho$. Thus by an isotopy through nearly straight fields, we may assume that our given field is obtained by upwards rotation of a perpendicular field.

We now apply the proof of the Local Compression Theorem [14; 4.4] to $\alpha$ and watch the effect on $\beta$. We observe that all the moves on $\alpha$ in the proof are small general position moves except one—localisation. But by definition localisation, if it moves $\alpha$ at all, moves it upwards to some extent (ie, increases its vertical component). It can then be seen that the effect on $\beta$ is also to move it upwards in the same sense. Thus $\beta$ moves through nearly straight fields. Figure 1 illustrates the case when $\alpha$ is on the downset $D$ (and hence $\beta = \rho$) and $\beta$ is unchanged by localisation.

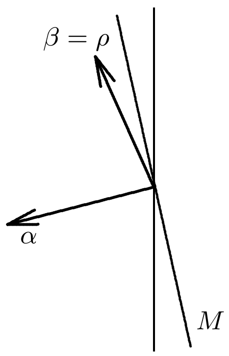


Figure 1

In general you can imagine localisation rotating $\alpha$ around (and upwards) in the plane perpendicular to $M$ and $\beta$ following the movement on a small cone around $M$.

It is now clear that the field used to generate the isotopy which straightens $\beta$ is roughly as nearly straight as the original field and therefore the whole process leaves the subsequent fields nearly straight.

Finally, we need to prove that $TF$ is homotopic to $G$ through bundle monomorphisms. But the construction provides a homotopy through bundle monomorphisms of $TM$ in $T(Q \times \mathbb{R}^n) \cong TQ \oplus T\mathbb{R}^n$ and moreover, by considering the $n$ normal fields, we actually have a homotopy through bundle monomorphisms $h$, say, of $TM \oplus T\mathbb{R}^n$ in $TQ \oplus T\mathbb{R}^n$ which starts with $G \oplus \mathrm{id}$ and ends with a monomorphism $J$ say, which is the derivative of the final parallel embedding extended by the normal fields. Thus $J|T\mathbb{R}^n = \mathrm{id}$ and projection on $Q$ carries $J$ to $TF$. But since the $n$ normal fields stay close to $T\mathbb{R}^n$, $h|T\mathbb{R}^n$ can be canonically homotoped to the constant homotopy. Extend this canonical homotopy to a homotopy of $h$ by choosing a complement for the image and the result is a homotopy $h'$ between $G \oplus \mathrm{id}$ and $J$ which is fixed on $T\mathbb{R}^n$. Projecting $h'$ on $TQ$ gives the required homotopy between $G$ and $TF$. □

**Addenda**

(1) (Relative version) *Let $K_0, M_0$ be submanifolds of $K, M$ of codimension 0. Suppose given a $K_0$–family $F$ of immersions of $M$ in $Q$ with extension (also denoted $F$) to a $K$–family of immersions of $M_0$ in $Q$. Suppose that $TF$ extends to a $K$–family $G$ of bundle monomorphisms of $TM$ in $TQ$. Then $F$ extends to a $K$–family $F^+$ of immersions of $M$ in $Q$ such that $TF^+$ is homotopic through $K$–families of bundle monomorphisms to $G$ and the homotopy is fixed where $TF$ is defined (and equal to $G$).*

(2) (Codimension zero version) *Addendum (i) holds if $q - m \geq 0$ and each component of each component of $M - M_0$ has relative boundary not in $M_0$.*

**Proof** We first observe that if $f\colon M \to Q$ is an immersion and we apply the proof of 2.3 to $Tf$ then the immersion constructed is precisely $f$. This is because the rotation of fibres of $TM$ into $M$ is in fact through less than a right-angle and the nearly straight fields constructed in 2.3 are actually straight. Moreover the homotopy constructed at the end of the proof is the constant homotopy in this case. Addendum (1) now follows from the main proof and addendum (2) follows by a handle decomposition argument as in addendum (ii) to the Local Compression Theorem [14; 4.4]. □

**Corollary 2.4** (Immersion theory) *Let $f\colon M_0 \to Q$ be an immersion then differentiation $T$ induces a weak homotopy equivalence between the spaces of immersions of $M$ in $Q$ equal to $f$ on $M_0$ and of bundle monomorphisms from $TM$ to $TQ$ equal to $Tf$ on $TM_0$.*

**Proof** Apply addendum (1) to 2.3 with $K = S^n$ to see that $T_*$ is surjective on $\pi_n$ and again with $K = S^n \times \mathbb{R}$ and $K_0 = S^n \times ((-\infty, 0] \cup [1, \infty))$ to see that $T_*$ is injective. □

The corollary yields the usual classification of immersions: regular homotopy classes of immersions of $M$ in $Q$ are in bijection with homotopy classes of tangential maps $TM$ to $TQ$. Again the proof is explicit. Given two immersions and a homotopy between the tangential maps the proof constructs a regular homotopy between the given immersions. Thus it can be used, for example, to describe an explicit way to turn the 2–sphere inside out in $\mathbb{R}^3$ and we intend to give such an explicit description in a separate paper. In this description we start from an explicit homotopy of bundle monomorphisms and an explicit embedding (the usual one) in $\mathbb{R}^3$. Then the proof provides a sequence of three flows which integrate to give the regular homotopy which turns the sphere inside out.

## 3 Loops–suspension theorem

We next show that the compression theorem can be used to give a short new proof of the classical result of James [6] on the homotopy type of loops–suspension and of the generalisation due to May [11] and Segal [16] and implicit in Milgram [12]. In [13] the arguments in this section are extended to both the equivariant case and the disconnected case (where group completions are needed).

We denote the free topological monoid on a based space $X$ by $X_\infty$ and denote the loop space on and suspension of $X$ by $\Omega(X)$, $S(X)$, respectively. We assume spaces are compactly generated and Hausdorff and we assume base points non-degenerate. In particular this means (up to homotopy) we can assume based spaces have *whiskers*, ie, the base point has a neighbourhood homeomorphic to $\{1\}$ in the interval $[0, 1]$. It will be convenient to identify the whisker with $[0, 1]$.

There is a map $k_X\colon X_\infty \to \Omega S(X)$ defined in [6]. Briefly, what $k_X$ does is to map the word $x_1 \cdots x_m$ to a loop in $S(X)$ which comprises $m$ vertical loops passing through $x_1, \ldots, x_m$ respectively, with the time parameters carefully adjusted to make the time spent on a subloop go to zero as the corresponding point $x_i$ moves to the basepoint of $X$.

**Theorem 3.1** *Let $X$ be path-connected. Then $k_X\colon X_\infty \to \Omega S(X)$ is a weak homotopy equivalence.*

**Proof** There is a well-known equivalent definition of $X_\infty$ (up to homotopy type) as the configuration space $C_1(X)$ of points in $\mathbb{R}^1$ labelled in $X$, of which we briefly recall the definition. Consider finite subsets of $\mathbb{R}^1$ labelled by points of $X$. An equivalence relation on such subsets is generated by deleting points labelled by the basepoint. $C_1(X)$ is the set of equivalence classes. The topology is induced from the topologies of $\mathbb{R}^1$ and $X$.

We shall give a geometric description for the homomorphism $\pi_n(C_1X) \to \pi_n(\Omega SX)$ induced by $k_X$. The construction is similar to a construction in [9], which used transversality to the base of a Thom space. Here we use transversality to $X$ in $X \times \mathbb{R}$ and to an interior point of the whisker in $X$.

We start by giving a geometric description for a map into $C_1(X)$. Let $Q$ be a smooth manifold and $f\colon Q \to C_1(X)$ a map. Then $f$ determines a subset of $Q \times \mathbb{R}^1$, continuously labelled by $X$, such that at points labelled in $X - \{*\}$ the projection on $Q$ is a local homeomorphism. This subset is determined up to introduction and deletion of points labelled by $*$. By approximating the local map to $\mathbb{R}^1$ by a smooth map, we can assume, by a small homotopy of $f$, that the subset of points labelled in $X - \{*\}$ is a smooth submanifold of $Q \times \mathbb{R}$. Further, by using transversality of the labelling map to an interior point $\frac{1}{2}$ in the whisker in $X$, and composing with the stretch $[0, \frac{1}{2}] \to [0, 1]$, we may assume that this labelled subset is in fact a smooth submanifold $W$ with boundary, such that the boundary is the subset labelled by $*$ and such that the projection on $Q$ is a local embedding (of codimension 0). Conversely, such a subset determines a map $f\colon Q \to C_1(X)$. Notice that $W$ is canonically framed in $Q \times \mathbb{R}^1$ by the $\mathbb{R}^1$–coordinate and also notice that if $Q$ and $f$ are based then $W$ can be assumed to be empty over the basepoint of $Q$.

Next we interpret maps in $\Omega SX$. A map $Q \to \Omega SX$ determines a map $f\colon Q \times \mathbb{R}^1 \to SX$. By making $f$ transverse to the suspension line through $\frac{1}{2}$ in the whisker in $X$, and composing with the stretch $[0, \frac{1}{2}] \to [0, 1]$, we may assume that $\overline{f^{-1}(SX - \{*\})}$ is a codimension 0 submanifold whose boundary maps to $*_{SX}$. By further making $f$ transverse to $X \times \{0\}$ we may assume that $f^{-1}((X - \{*\}) \times \{0\})$ is a framed codimension 1 submanifold $W$ with boundary, equipped with a map $l\colon W \to X$ such that $l^{-1}* = \partial W$ and that $f$ maps framing lines to suspension lines and the rest is mapped to $*_{SX}$. Conversely such a framed submanifold determines a map $Q \to \Omega SX$. Further if $Q$ and $f$ are based then $W$ is empty over the basepoint of $Q$.

With these geometric descriptions there is an obvious forgetful map $[Q, C_1X] \to [Q, \Omega SX]$ which may be seen to be induced by $k_X$. Now consider the case when $Q = S^n$ and consider a framed manifold $W$ representing an element of $\pi_n(SX)$. If it has a closed component we can change the labelling function to map a small disc to $*_X$ by using a collar on the disc and mapping the collar lines to a path to the basepoint in $X$. Then the interior of the disc, which is now labelled by $*_X$, may be deleted from $W$. After eliminating all closed components in this way the compression theorem (the codimension 0 case) implies that $\pi_n(C_1X) \to \pi_n(\Omega SX)$ is surjective.

Injectivity is proved similarly by using the case $Q = S^n \times I$ and working relative to $Q \times \{0,1\}$. $\square$

Notice that only the Global Compression Theorem [14; 2.1] (and addendum (i)) were used for the proof of 3.1 so the complete new proof is short. However, for the full loops–suspension theorem below we need the multi-compression theorem. (For the shortest proof of the full theorem, use the short proof of the multi-compression theorem given in [15].)

Let $C_n(X)$ denote the configuration space of points in $\mathbb{R}^n$ labelled in $X$, with, as before, points labelled by $*$ removable. There is a map $q_X\colon C_n(X) \to \Omega^n S^n(X)$ defined in a similar way to $k_X$ (as interpreted in the last proof) using little cubes around the points in $\mathbb{R}^n$ with axes parallel to the axes of $\mathbb{R}^n$.

**Theorem 3.2** *Let $X$ be a path-connected topological space with a non-degenerate basepoint then $q_X$ is a weak homotopy equivalence.*

**Proof** The proof is similar to the proof for theorem 3.1. A map $Q \to C_n(X)$ can be see as partial cover of $Q$ embedded in $Q \times \mathbb{R}^n$ and a map $Q \to \Omega^n S^n(X)$ can be see as a framed codimension $n$ submanifold of $Q \times \mathbb{R}^n$.

There is then a function $\text{Maps}(Q \to C_n(X))$ to $\text{Maps}(Q \to \Omega^n S^n(X))$ given by taking the parallel framing on the partial cover. This can be seen to be given by composition with $q_X$. Closed components in the submanifold of $Q \times \mathbb{R}^n$ may be punctured as in the last proof. The codimension 0 case of the multi-compression theorem now implies that $q_X$ induces a bijection between the sets of homotopy classes. $\square$

# 4 Embeddings and knots

Two basic problems of differential topology are the *embedding problem* : given two manifolds $M$ and $Q$, decide whether $M$ embeds in $Q$, and the *knot problem* : classify embeddings of $M$ in $Q$ up to isotopy. The corresponding problems for immersions (replace embedding by immersion and isotopy by regular homotopy) are, in some sense, solved by immersion theory; ie, solved by reducing to vector bundle problems for which there are standard obstruction theories. Now if $M$ embeds in $Q$ then it certainly immerses in $Q$ and if two embeddings are isotopic then they are certainly regularly homotopic. Thus it makes sense to consider the *relative embedding problem* : decide whether a given immersion of $M$ in $Q$ is regularly homotopic to an embedding, and the *relative knot problem* : given a regular homotopy between embeddings, decide if it can be deformed to an isotopy. Since a manifold often immerses in a considerably lower dimension than that in which it embeds, it makes sense to consider the following more general problems.

**Embedding problem 4.1** *Suppose given an immersion $f\colon M \to Q$ and an integer $n \geq 0$. Decide whether $f \times 0\colon M \to Q \times \mathbb{R}^n$ is regularly homotopic to an embedding.*

**Knot problem 4.2** *Classify, up to isotopy, embeddings of $M$ in $Q \times \mathbb{R}^n$ within the regular homotopy class of $f \times 0\colon M \to Q \times \mathbb{R}^n$, where $f\colon M \to Q$ is a given immersion.*

The compression theorem gives substantial information on these problems. In particular we can give formal solutions which make it easy to define obstructions to the existence of such embeddings or isotopies.

### The embedding theorem

Turning first to the embedding problem. Suppose that we are given an immersion $f\colon M \to Q$ and suppose that it is self-transverse [10]. Then there is a simple obstruction theory to decide if it is covered by an embedding in $Q \times \mathbb{R}^n$. Construct a neighbourhood system $N$ by choosing coherent tubular neighbourhoods on each stratum of the image. This neighbourhood system comprises a number of disc bundles each with a collection of subdisc bundles. The embedding is covered by an embedding in $Q \times \mathbb{R}^n$ if and only if this neighbourhood system admits a particular kind of structure described precisely by Koschorke and Sanderson [9] as follows.

There are classifying spaces $\mathcal{I}^c$ and $\mathcal{I}^c_n$ with a natural map $p\colon \mathcal{I}^c_n \to \mathcal{I}^c$, where $c = q - m$ (codimension). The space $\mathcal{I}^c$ classifies self-transverse immersions of codimension $c$ and $\mathcal{I}^c_n$ classifies such immersions covered by an embedding in codimension $c + n$. More precisely, $\mathcal{I}^c$ and $\mathcal{I}^c_n$ have good geometric structures and transversality can be defined for maps of manifolds. Any map can be made transverse and a transverse map $Q \to \mathcal{I}^c$ determines a self-transverse immersed submanifold of $Q$ of codimension $c$ and equipped with a neighbourhood system and conversely (see also [3; sections 2 and 4] for a detailed discussion of a special case). Thus the immersion of $M$ in $Q$ and the neighbourhood system $N$ determine a transverse map $\alpha_f\colon Q \to \mathcal{I}^c$; the immersion is covered by an embedding in $Q \times \mathbb{R}^n$ if and only if this map lifts to $\mathcal{I}^c_n$. These spaces have convenient descriptions as configuration spaces; $\mathcal{I}^c$ can be identified with $C_\infty(MO_c)$, the configuration space of points in $\mathbb{R}^\infty$ labelled in the Thom space of the universal $c$–dimensional vector bundle. Similarly $\mathcal{I}^c_n$ can be identified with $C_n(MO_c)$, the configuration space of points in $\mathbb{R}^n$ labelled in the Thom space. Then $p$ is induced by the inclusion $\mathbb{R}^n \subset \mathbb{R}^\infty$.

There are further classifying spaces $\mathcal{V}^c$ and $\mathcal{V}^c_n$ which classify neighbourhood systems of the types encountered above. Thus we have a transverse map $\beta_f\colon M \to \mathcal{V}^c$ classifying the neighbourhood system $N$ and the immersion is covered by an embedding in $N \times \mathbb{R}^n$ if and only if $\beta_f$ lifts to $\mathcal{V}^c_n$. These spaces also have descriptions as configuration spaces; $\mathcal{V}^c$ and $\mathcal{V}^c_n$ can be identified with $C^\bullet_\infty(MO_c)$ and $C^\bullet_n(MO_c)$ respectively, which are the spaces of based configurations in $\mathbb{R}^\infty$ and $\mathbb{R}^n$ respectively, labelled as before in $MO_c$ and such that the basepoint is labelled in the base $BO_c$. There are natural maps $\mathcal{V}^c \to \mathcal{I}^c$ and $\mathcal{V}^c_n \to \mathcal{I}^c_n$ given by forgetting the basing; there is also a natural map $p_0\colon \mathcal{V}^c_n \to \mathcal{V}^c$ induced by inclusion $\mathbb{R}^n \subset \mathbb{R}^\infty$ and finally a natural map $\mathcal{V}^c \to BO_c$ given by considering the base label. The latter map classifies the induced tubular neighbourhood of the immersion.

The given immersion now determines the following pull-back square in which $\alpha_f$ and $\beta_f$ are transverse :

$$\gamma_f\colon \quad \begin{array}{ccc} M & \xrightarrow{\beta_f} & \mathcal{V}^c \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathrm{nat}} \\ Q & \xrightarrow{\alpha_f} & \mathcal{I}^c \end{array}$$

We call a homotopy between such pull-back squares a *pull-back homotopy* if the homotopy determines a pull-back square of the following form, where $\alpha_F$ and $\beta_F$ are transverse :

$$\begin{array}{ccc} M \times I & \xrightarrow{\beta_F} & \mathcal{V}^c \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle \mathrm{nat}} \\ Q \times I & \xrightarrow{\alpha_F} & \mathcal{I}^c \end{array}$$

**Embedding theorem 4.3** *Suppose we are given a self transverse immersion* $f\colon M \to Q$ *and that* $c = q - m \geq 1$. *Then the immersion* $f \times 0\colon M \to Q \times \mathbb{R}^n$

*is regularly homotopic to an embedding if and only if the pull-back square $\gamma_f$ is homotopic by a pull-back homotopy to a square which lifts over the natural maps $p, p_0$ to give a square of the form :*

$$\begin{array}{ccc} M & \xrightarrow{\beta_g} & \mathcal{V}^c_n \\ g \downarrow & & \downarrow \text{nat} \\ Q & \xrightarrow{\alpha_g} & \mathcal{I}^c_n. \end{array}$$

**Proof** By the classification properties of $\mathcal{I}^c$ and $\mathcal{V}^c$, a pull-back homotopy is equivalent to a self-transverse regular homotopy of $M$ in $Q$. But any regular homotopy between self-transverse maps can be made self-transverse, so a pull-back homotopy is equivalent to a regular homotopy of $M$ in $Q$.

Suppose $f \times 0 \colon M \to Q \times \mathbb{R}^n$ is regularly homotopic to an embedding. Let $F\colon M \times I \to Q \times \mathbb{R}^n \times I$ be determined by this regular homotopy. Thus $F_0 = F \,|\, (M \times \{0\}) = f \times 0$ and $F_1 = F \,|\, (M \times \{1\})$ is an embedding. The canonical $n$–frame on $F_0$ extends to an $n$–frame on $F$. Apply the multi-compression theorem to $F_1$ to yield an isotopy of $F_1$ to $F_1'$ which compresses to an immersion $g\colon M \to Q$. Extend the isotopy of $F_1$ to an isotopy of $F$ to $F'$ say rel $F_0$ and apply the multi-compression theorem again to compress $F'$ to a regular homotopy between $f$ and $g$. By the remarks made above, there is there is a pull-back homotopy of the required form. The converse is clear. □

The natural map $\mathcal{V}^c \to \mathcal{I}^c$ is not a fibration. Thus the condition of pull-back homotopy is difficult to interpret homotopy theoretically, and the embedding theorem does not immediately reduce the problem to an obstruction theory. However it does readily lead to many obstructions to the existence of embeddings. Any algebraic topological invariant of $\mathcal{I}^c$ which does not come from $\mathcal{I}^c_n$ or of $\mathcal{V}^c$ which does not come from $\mathcal{V}^c_n$ is such an obstruction. The former obstructions are obstructions to a regular cobordism to an embedding, and were considered in [9]. The latter are obstructions to deforming the neighbourhood system for $M$. There are also obstructions coming from the combinatorial structure of the classifying spaces analogous to the (generalised) James–Hopf invariants of a □–set, defined in [3].

### Classification of knots

There is a similar analysis for the knot problem. Rather than continuing to consider embeddings in $Q \times \mathbb{R}^n$ which cover immersions in $Q$, we shall consider embeddings equipped with $n$ independent normal vector fields (for example framed embeddings) since by the compression theorem these are equivalent. Such an embedding $f$ determines a pull-back square :

$$\rho_f\colon \quad \begin{array}{ccc} M & \xrightarrow{\beta_{f_1}} & \mathcal{V}_n^c \\ f_1 \downarrow & & \downarrow \text{nat} \\ Q & \xrightarrow{\alpha_{f_1}} & \mathcal{I}_n^c. \end{array}$$

where $f_1 = \text{proj} \circ f$.

The arguments used to prove theorem 4.3 now prove :

**Theorem 4.4** *Suppose that $f, g\colon M \to Q \times \mathbb{R}^n$ are embeddings equipped with $n$ independent normal vector fields and that $q - m \geq 1$. Then $f$ is isotopic to $g$ if and only if $\rho_f$ is homotopic to $\rho_g$ by a pull-back homotopy.* □

**Corollary 4.5** Classification of knots *Suppose that $q - m \geq 1$. There is a bijection between isotopy classes of embeddings $f$ of $M$ in $Q \times \mathbb{R}^n$ equipped with $n$ independent normal vector fields and pull-back homotopy classes of squares $\rho_f$.* □

The corollary gives many knot invariants, for example any homotopy invariant of $\mathcal{I}_n^c$, pulled back to $Q$, or of $\mathcal{V}_n^c$, pulled back to $M$. The former are invariants of the cobordism class of the knot and the latter are invariants of the neighbourhood system under cobordism through neighbourhood systems of $M$. There are also the combinatorial invariants mentioned earlier.

**Remarks** The invariants and obstructions discussed above have strong connections with many existing invariants. The case $c = 1, n = 1$ is studied by Fenn, Rourke and Sanderson see in particular [3; sections 2 and 4], where classifying spaces related to $\mathcal{I}_1^1$ but depending of the fundamental rack, are also considered. There are combinatorial invariants defined in this case, for example the generalised James–Hopf invariants. These link with those defined by Koschorke and Sanderson [9], which as indicated above is an analysis of the problems considered here but only up to cobordism.

We shall give more details here and also explain connections with other known obstructions and invariants in a subsequent paper.

# 5 Controlling singularities of a projection

**Definition** A *weakly stratified set* is a set $X$ with a flag of closed subsets

$$X = S_0 \supset S_1 \supset S_2 \ldots \supset S_t \supset S_{t+1} = \emptyset$$

such that, for each $i = 0, \ldots, t$, $S_i - S_{i+1}$ is a manifold.

We also say that $S_i$ is a *weak stratification* of $X$ and we call the manifolds $S_i - S_{i+1}$ the *strata*.

**Remark** This is very much weaker than the usual notion of a stratified set — there is no condition on the neighbourhood of $S_{i+1}$ in $S_i$ or any relationship between the dimensions of the strata.

**Definitions** Suppose that $X$ is a weakly stratified set and that $X \subset W$ (a manifold). Suppose that $\xi^n$ is a plane field on $W$ (ie a $n$–subbundle of $TW$) defined at $X$. We say $\xi$ is *weakly normal* to $X$ if $\xi$ is normal to $S_i - S_{i+1}$ for each $i = 0, \ldots, p$.

Suppose that $M^m \subset W^w$ and that $\xi^n$ is a plane field defined at $M$. We say that $\xi$ has *regular singularities on $M$* if it is normal to a weak stratification of $M$.

**Example 5.1** A plane field in general position has regular singularities. However so do many plane fields which are far from general position. Here is an explicit example with $n = 1$ constructed by Varley [19]. Let $\alpha$ be the $x$–axis in $\mathbb{R}^3$ and $C \subset \alpha$ a cantor set. Let $\pi$ be the surface (a smooth plane) given by $z = y^3$ which contains $\alpha$ and has a line of inflection along $\alpha$. Let $\xi$ be the line field parallel to the $y$–axis and distort $\xi$ to have a small negative $z$–component off $C$. Then $\xi$ is tangent to $\pi$ precisely at $C$ and very far from general position. But it is has regular singularities by choosing $\alpha$ as one stratum and $\pi$ as the next.

It is easy to construct plane fields with non-regular singularities: for an example with $n = 1$, choose any line field for which part of a flow line lies in $M$.

$C^0$**–Singularity Theorem 5.2** *Suppose that $M^m \subset W^w$ and that $\xi$ is an integrable $n$–plane field on $TW$ defined on a neighbourhood $U$ of $M$ such that $\xi$ has regular singularities on $M$ and that $n + m < w$. Suppose given $\varepsilon > 0$ and a homotopy of $\xi$ through integrable plane fields on $TW$ defined on $U$ finishing with the plane field $\xi'$. Then there is an ambient isotopy of $M$ in $W$ which moves points at most $\varepsilon$ moving the pair $M, \xi|M$ to $M', \xi'|M'$.*

The theorem gives an answer to the $C^0$–Singularity Problem (stated in section 1) in the case that $\dim(M) < \dim(Q)$. Take $\xi'$ to be the tangent bundle to the fibres of $p$ then the singularities of $p|M$ can be made to coincide with those of any plane field homotopic to $\xi'$ with regular singularities.

Some condition on the singularities is clearly necessary, for example, again with $n = 1$, suppose that part of a flow line of the normal line field lies in $M$ and is not already vertical (thinking of $\xi'$ as vertical) then no *small* isotopy can make this field vertical. The condition of regularity is very weak as can be seen by considering examples similar to 5.1. The condition that the plane fields are *integrable* is needed for our proof, but we do not have an example to show that it is necessary for the result. Note that example 5.1 is integrable and see [19] for more examples.

**Proof** We shall prove a more general result: $M$ is replaced by any weakly stratified set $X$ such that dimensions of strata are $< w - n$ and $\xi$ is weakly normal to $X$. The idea is to apply the Normal Deformation Theorem to each stratum in turn starting with $S_t$ and continuing with $S_{t-1} - S_t$ etc). After the $t - i^{\text{th}}$ move $\xi$ concides with $\xi'$ on $S_i$ and with care this can also be assumed to be true near $S_i$ and in particular in a neighbourhood of $S_i$ in $S_{i-1}$. The next move is made relative to a smaller neighbourhood, and the result is proved in $t + 1$ steps.

So the only point that needs work is the point that $\xi$ can be assumed to concide with $\xi'$ *near* $S_i$. This is where integrability is needed. By integrability we can assume that $\xi$ is the tangent bundle to a foliation $\mathcal{F}$ of $W$ defined near $S_i$ and similarly $\xi'$ is the tangent bundle to $\mathcal{F}'$. Now $\mathcal{F}$ and $\mathcal{F}'$ coincide at $S_i$ and by a $C^\infty$–small isotopy $\mathcal{F}$ can be moved to $\mathcal{F}'$ near $S_i$ and this carries $\xi$ into coincidence with $\xi'$ near $S_i$ as required. □

**Addenda** The proof works (indeed was given) for a weakly stratified set instead of a submanifold and has a natural relative version directly from the proof. There is also a parametrised version which follows by combining the proof of the parametrised Normal Deformation Theorem [14, page 425] or [15, page 439] with the last proof:

*Suppose that we have a family of embeddings of $M_t$ in $W$ where $t \in K$ (a parameter manifold) together with integrable plane fields $\xi_t$ for $t \in K$ having regular singularities with $M_t$. Suppose further that the singularities are "locally constant" over $K$ (ie the whole situation is locally trivial). Suppose given a $K$–parameter homotopy of $\xi_t$ through integrable plane fields to $\xi'_t$. Then there is a $K$–parameter family of small ambient isotopies carrying $M_t, \xi_t$ to $M'_t, \xi'_t$.*

Varley [19] gives a more general parametrised theorem in which the signularities are not assumed locally constant.

**Comments** The philosophy of our solution to the $C^0$–singularity problem is that the singularities of an arbitrary plane field are instrinsic to the embedding $M \subset W$ and invariant under isotopy of the situation. There is a very nice description in the metastable range where there is a single singularity manifold (in general position). This can be regarded as a fine bordism class in the sense of Koschorke [7] and is an instrinsic embedded characteristic class. For more detail see [19].

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