WHAT IS A WELDED LINK? corrected version (2018)

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We give a geometric description of welded links in the spirit of Kuperberg's description of virtual links: "What is a virtual link?" [8].

Keywords: Welded link; Classical link; Virtual link; Toral surface

1. Intoduction

Welded links were introduced by Fenn, Rimanyi and Rourke in their 1997 Topology paper [2]. In 1999 Kaufmann [7] introduced virtual links. The two concepts are closely connected. Welded links correspond to framed virtual links with one of the two so-called "forbidden moves" allowed.

In 2003, in a short elegant paper, "What is a virtual link?", Kuperberg [8] gave a good geometric description of virtual links: they correspond to links in oriented surfaces cross I and each link has a unique representative with the surface having minimal genus.

The purpose of this note is to investigate similar descriptions for welded links. There is no obvious way to extend Kuperberg's result to welded links, but using ideas of Satoh [11] we can give a satisfactory 4–dimensional interpretation.

Author's note This is the second version of this paper. The first version (2006) published in the proceedings [13] contained a systematic error. Welded links are intrinsically framed objects belonging to the world of *racks* rather than *quandles*. This fact, although well known to me, was overlooked in the 2006 version; indeed the Reidemeister R_1 moves were listed as allowable for welded links, which they are not. In this version, this is corrected and both classical and virtual links are all framed to match. Apart from these changes the statement of the main theorem (4.1) has been tightened.

It was possible to misunderstand the original version. The context is fibred embeddings and isotopies and these have local triviality conditions as is usual for fibred objects. These were not spelt out in the original version and the theorem was false as stated if interpreted literally.

I am grateful to the authors of [14] for drawing my attention to these errors.

There is a problem with terminology. The fact that welded links are intrinsically framed is not well known. Authors writing after the originators in [2] have often assumed that welded links are unframed and have allowable R_1 moves. Indeed it was this change of usage which misled me into making the error in the original version. To avoid this problem in the future, I suggest that welded links in their original definition should now be referred to as *framed welded links* and welded links in which R_1 moves are allowed should be referred to as *unframed welded links*. In this paper all welded links are framed.

2. Four classes of links

We define the various kinds of links that we consider by describing the corresponding allowable moves. This is done in Fig. 1.

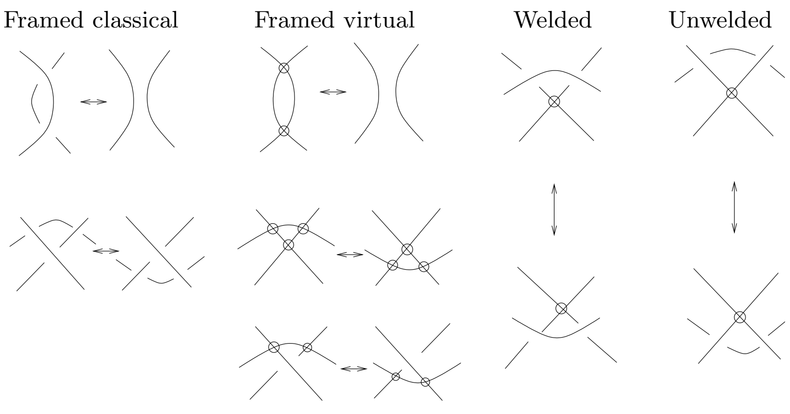


Fig. 1. Allowable moves

Framed classical links are defined by the familiar Reidemeister R_2 and R_3 moves. Across the figure come, in order, framed virtual links, welded links and unwelded links. The moves are cumulative from left to right, so that all the moves are allowed for an unwelded link. The new move allowed for welded links and the further move allowed for unwelded links are known as the "forbidden moves" in virtual knot theory. The first move (the one allowed for welded links) suggests that the virtual crossing is fixed down on the page and this is why we called these objects "welded". The second suggests that the weld has come free and this is why I suggest that these be called "unwelded".

The definitions make it clear that there are natural maps:

FRAMED CLASSICAL $\rightarrow$ FRAMED VIRTUAL $\rightarrow$ WELDED $\rightarrow$ UNWELDED

Where for example FRAMED CLASSICAL means equivalence classes of framed classical links. There is a similar diagram for knots. If R_1 moves are allowed then by Nelson [9] or Kanenobu [6] an unwelded knot is always trivial. But if R_1 moves are not allowed (as in this paper) then there are two writhe invariants: one of these is the usual writhe in $\mathbb{Z}$ coming from counting ordinary crossings with sign and the other in $\mathbb{Z}/2$ given by counting welded crossings mod 2. For unwelded links the situation is more complicated because there are also invariant linking numbers. Because of the welded crossings the number of over and under crossings of one component with another need not be equal and there are two independent linking numbers for a pair of components and therefore $2\binom{n}{2}$ linking numbers in total. Fig. 2 shows a framed virtual link of two components with the left linked once *under* the right and the right linked once *over* the left.

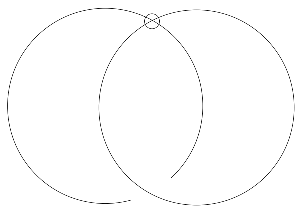


Fig. 2. Framed virtual Hopf link

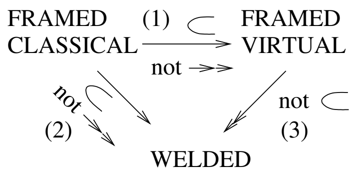
Using the methods of Nelson [9] or Kanenobu [6], it can be seen that these writhes and linking numbers form a complete set of invariants and we have the following theorem, of which a detailed proof (in the unframed case) has been given by Okabayashi [10; Theorem 8].

Theorem 2.1 (Classification of unwelded links). *Equivalence classes of unwelded links of n components form a group under connected sum isomorphic to $\mathbb{Z}^{n+2\binom{n}{2}} + (\mathbb{Z}/2)^n$.*

The relationships between the other three classes of links are as follows:

Theorem 2.2. (Relationship between the other classes of links).

(1) FRAMED CLASSICAL *embeds as a proper subset of* FRAMED VIRTUAL.
(2) FRAMED CLASSICAL *embeds as a proper subset of* WELDED.
(3) FRAMED VIRTUAL *maps onto but is not isomorphic to* WELDED.



Remark 2.1. (2) answers a question of Paulo Bellingeri comunicated to me by Kamada [5].

3. Proof of theorem 2.2

(1) follows from (2) and (3): since FRAMED CLASSICAL embeds in WELDED it must embed in FRAMED VIRTUAL, and if FRAMED CLASSICAL mapped onto FRAMED VIRTUAL then since FRAMED VIRTUAL maps onto WELDED then it would map onto WELDED.

It is clear that FRAMED VIRTUAL maps onto WELDED, so to prove (3) it suffices to give an example of a non-trivial virtual link which is trivial as a welded link. Look at the virtual reef knot in Fig. 3.

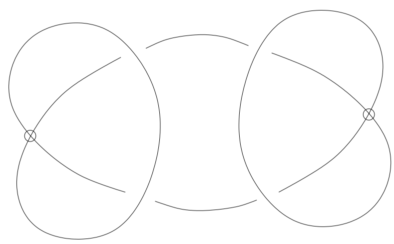


Fig. 3. Virtual reef knot

It is clear that it is trivial as a welded knot since the two loops can be pulled over the welded crossings. To see that it is non-trivial as a virtual knot, consider the reflection in a horizontal plane—the welded reef knot in

Fig. 4. This is non-trivial as a welded knot, because the fundamental group (which along with the fundamental rack is defined for welded links [2]) is not $\mathbb{Z}$. You can read this from the diagram. Both the rack and the group have presentation $\{a, b \mid a^b = b^a\}$. As a group this is the Baumslag–Solitar group $B(2, 1)$. To see that this group is non-abelian add the relators $a^2 = b^2 = 1$, then a quick computation shows that $(ab)^3 = 1$ and the resulting group is the symmetric group S_3.

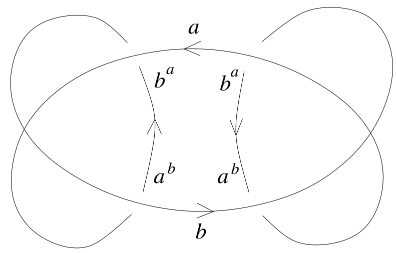


Fig. 4. Welded reef knot

If the virtual reef knot were trivial as a virtual link, then by symmetry so would the welded reef knot (virtual moves are symmetric under change of crossing) and therefore it would be trivial as a welded link.

Now the Baumslag–Solitar group $B(2, 1)$ is not a 3–manifold group (Heil [4], see also Shalen [12]), and hence there is no classical link which is equivalent to it, proving that FRAMED CLASSICAL does not map onto WELDED.

The only part of the theorem left to prove is that FRAMED CLASSICAL embeds in WELDED. This follows from the fact already mentioned that the fundamental rack of a framed classical link extends to welded links. The basic classification theorem for links in terms of racks (see eg, [1; Theorem 5.6]) says that the fundamental rack is a complete invariant for framed classical links. Therefore if two framed classical links are equivalent as welded links then their fundamental racks are isomorphic and they are equivalent as framed classical links as required.

4. Toral surfaces

As we have seen, there are welded knots whose fundamental group is not a 3–manifold group. It follows that we cannot find a representation of welded links as links in 3–manifolds with a corresponding fundamental group. This is also true for virtual links, where we have Kuperberg's representation. It turns out that the fundamental group in this case has the interpretation of the *reduced fundamental group*—the fundamental group of $S \times I/S \times \{0\}$ (this follows from the corresponding statement for racks, see [3; Lemma 3.3 and below Thorem 4.5]).

It is possible that the Kuperberg representation could give a similar representation for welded links. The relevant question is this:

Question 4.1. Is the minimal representation of a welded links (amongst virtual lifts) unique?

The evidence is that the answer is "No". The forbidden move changes the virtual link drastically and it is unlikely that the corresponding minimal representations will be the same. However I do not have a specific example of this.

But if we move up to four dimensions then we can represent a given virtual or welded link by an embedding of a number of tori and moreover the fundamental group is correct without the need for any reduction. The relevant idea here is due to Satoh [11]. Satoh gives a representation, and with a small modification his construction can be used to give a classification using *toral surfaces*.

Think of $\mathbb{R}^4$ as the product $\mathbb{R}^2 \times \mathbb{R}^2$ equipped with the *projection* $p\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2 \times \{0\}$. Accordingly call the subsets $\{x\} \times \mathbb{R}^2$ the *fibres* of $\mathbb{R}^4$.

By a *torus* we mean a copy of $S^1 \times S^1$ and we call the subsets $\{x\} \times S^1$, S^1*–fibres* and the first factor the *base*. A *toral surface* is a collection of tori embedded *fibrewise* in $\mathbb{R}^4$. In other words each S^1–fibre of each torus lies in a fibre of $\mathbb{R}^4$ and the collection is locally trivial: there is a neighbourhood of each point in the domain on which the embedding is equivalent over the induced map $C \to \mathbb{R}^2 \times \{0\}$ to the product with an interval included in a fibre of $\mathbb{R}^4$ where C is the union of the bases in the collection.

Each fibre of $\mathbb{R}^4$ contains a number of S^1–fibres possibly arranged as in Fig. 5.

Theorem 4.1 (Classification of welded links). *There is a natural bijection between equivalence classes of welded links and fibrewise isotopy classes of toral surfaces. Here fibrewise means that the isotopy is through toral surfaces and the collection is locally trivial over* $C \times I \to \mathbb{R}^2 \times \{0\} \times I$.

5. Proof of theorem 4.1

The induced map $C \to \mathbb{R}^2 \times \{0\}$ gives a diagram in $\mathbb{R}^2$. By moving into general position, this diagram is immersed with a number of double points.

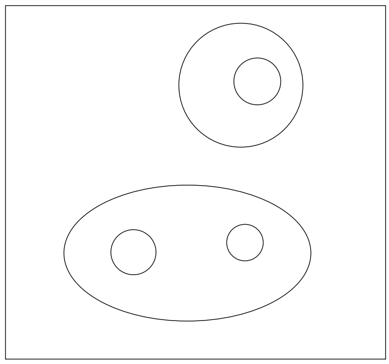
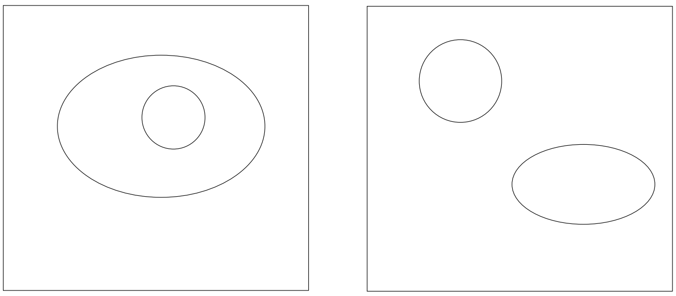


Fig. 5. A typical fibre

The fibre over each double point contains two S^1–fibres which can have one of two arrangements, drawn in Fig. 6. On the left, the circles are nested and we interpret this as a classical crossing in the diagram (with the outer circle corresponding to the upper strand) and on the right the circles are unnested and we interpret this as a virtual (or welded) crossing.

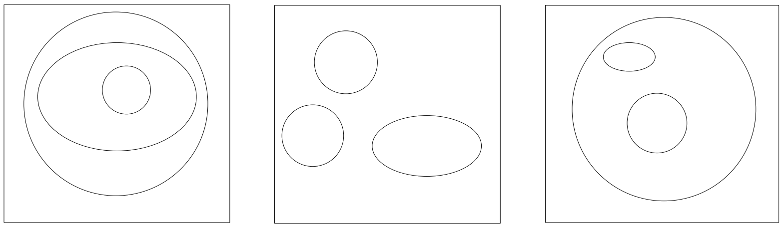


Classical Virtual

Fig. 6. Double points

Now consider an isotopy through toral surfaces. Again, using general position, the critical times correspond to standard Reidemeister moves on the diagram. The definition of fibrewise isotopy implies that the diagram moves by a regular homotopy and therefore there are no R_1 moves. (**Note** there was an error at this point in the 2006 version. It was incorrectly stated that there are R_1 moves.) There are two R_2 moves where two fibres (of the same type) merge into one-another and dissapear.

Turning now to R_3 moves, there is a triple point in the middle of the move and there are three corresponding arrangements of S^1–fibres drawn in Fig. 7.



Classical Virtual Welded

Fig. 7. Triple points

On the left, all three crossings are classical and this projects in $\mathbb{R}^2$ to a classical R_3 move. In the middle, all three are virtual and this projects to a virtual R_3 move. Finally, on the right, two crossings are classical and the third is virtual and this projects to the welded R_3 move (the first "forbidden move").

We have shown that a toral surface determines a welded knot and that a fibrewise isotopy determines an equivalence through allowable moves. The converse (the construction of a corresponding toral surface from a diagram and an isotopy from a sequence of moves) is a simple exercise.

Acknowledgements

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