A short exposition of Gödel's theorem

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We assume that we have a formal language and a faithful system for coding statements in that language by natural numbers. The precise details of the language and of the coding process are irrelevant but "statement" has no special meaning and can be taken to mean any string of symbols used by the language.

If X is a statement and $x \in N$ is its coded form then we say that x is the Gödel number of X and write $x = \mathrm{GN}(X)$.

We assume that "proof" has a formalisation in the language and that the coding used is also formalised, in other words:

$$X \text{ is provable} \qquad X \text{ is not provable} \qquad x = \mathrm{GN}(X)$$

are all formal statements.

Now consider the following statements:

$$\text{(R)} \qquad \begin{array}{l} X = X(t) \text{ is a statement involving a variable } t \\ x = \mathrm{GN}(X) \qquad X(x) \text{ is not provable} \end{array}$$

Here we have used functional notation with its usual meaning. $X(x)$ means $X(t)$ with x substituted for the variable t.

R denotes the collection of statements marked (R) and can itself be regarded as one formal statement in the formal language. Note that R involves x which we now regard as a variable, in other words $R = R(x)$.

Let $r = \mathrm{GN}(R)$ and let $G = R(r)$ ie the result of substituting r for x in R.

Now G says that $X(r)$ is not provable where $r = \mathrm{GN}(X)$. But if $r = \mathrm{GN}(X)$ then $X = R$ since $r = \mathrm{GN}(R)$, so G is equivalent to the statement that $R(r)$ is not provable.

In other words G is equivalent to "G is not provable".

Further it can be seen that the formal statement "G is equivalent to G is not provable" has a formal proof within the system.