Notes on de Sitter space

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Minkowski space

Minkowski n–space M^n is $\mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1}$ (time times space) equipped with the standard (pseudo)-metric of signature $(-, +, \ldots, +)$:

$$ds^2 = -dx_0^2 + dx_1^2 + \ldots + dx_{n-1}^2$$

The time coordinate is x_0 and the space coordinates are $x_1, \ldots, x_{n-1}$. For $x, y \in M^n$, the (pseudo-)inner product $\langle x, y\rangle$ is defined to be $-x_0y_0 + x_1y_1 + \ldots + x_{n-1}y_{n-1}$.

The *Lorentz n–group* is the group of "isometries" (transformations preserving the inner product) of Minkowski space, fixing $\mathbf{0}$, and preserving the time direction. This implies that a Lorentz transformation is an linear isomorphism of $\mathbb{R}^n$ as a vector space. If, in addition to preserving the time direction, we also preserve space orientation then the group can be denoted $\mathrm{SO}(1, n-1)$. Notice that a Lorentz transformation which preserves the x_0–axis is an othogonal transformation of the perpendicular $(n-1)$–space, thus $\mathrm{SO}(n-1)$ is a subgroup of $\mathrm{SO}(1, n-1)$ and we refer to elements of this subgroup as (Euclidean) rotations about the x_0–axis.

Minkowski 4–space is simply called *Minkowski space* and is the simplest example of a space-time. The Lorentz 4–group is called the *Lorentz group*.

Space-times

A pseudo-Riemannian manifold L is a manifold equipped with non-degenerate quadratic form g on its tangent bundle called the *metric*. A *space-time* is a pseudo-Riemannian 4–manifold equipped with a metric of signature $(-, +, +, +)$. Minkowski space is the simplest example of a space-time and in general the Lorentz gropup acts as structure group for the tangent bundle of a space-time. The metric is often written as ds^2, a symmetric quadratic expression in differential 1–forms as above. A tangent vector v

is *time-like* if $g(v) < 0$, *space-like* if $g(v) > 0$ and *null* if $g(v) = 0$. The set of null vectors at a point form the *light-cone* at that point and this is a cone on two copies of S^2. The set of time-like vectors at a point breaks into two components bounded by the two components of the light cone. A choice of one of these components determines the *future* at that point and we always assume *time orientability*, ie a global choice of future pointing light cones. An *observer field* on a space-time L is a smooth future-oriented time-like unit vector field on L.

de Sitter and hyperbolic spaces

Now go up one dimension. *Hyperbolic* 4*–space* is the subset

$$\mathbb{H}^4 = \{\langle x, x\rangle = -a^2,\ x_0 > 0 \mid x \in M^5\}.$$

de Sitter space is the subset

$$\mathrm{deS} = \{\langle x, x\rangle = a^2 \mid x \in M^5\}.$$

There is an isometric copy $\mathbb{H}^4_q$ of hyperbolic space with $x_0 < 0$. The induced metric on hyperbolic space is Riemannian and on de Sitter space is Lorentzian. Thus de Sitter space is a space-time. It is a solution of Einstein's equations with positive cosmological constant $\Lambda = 3/a^2$ and no matter.

The *light cone* is the subset

$$L = \{\langle x, x\rangle = 0 \mid x \in M^5\}.$$

and is the cone on two 3–spheres with natural conformal geometries (see hyperbolic geometry below). These are S^3 and S^3_q where S^3 is in the positive time direction and S^3_q negative.

Projective geometry

Points of

$$S^3 \cup S^3_q \cup \mathrm{deS} \cup \mathbb{H}^4 \cup \mathbb{H}^4_q \tag{1}$$

are in natural bijection with half-rays from the origin and we call this *half-ray space*. By considering full rays (lines) through the origin we get a copy of projective 4–space $\mathbb{P}^4$ which has half-ray space as its unique double cover. $\mathrm{SO}(1,4)$ acts faithfully on $\mathbb{P}^4$ by projective transformations. Each of $S^3, S^3_q, \mathbb{H}^4$ and $\mathbb{H}^4_q$ is faithfully represented as a subset of $\mathbb{P}^4$ with copies identified. $\mathrm{SO}(1,4)$ acts on $\mathbb{H}^4$ by hyperbolic isometries (see below). The natural linear structure on $\mathbb{P}^4$ (given by subspaces of $\mathbb{R}^5$) induces a natural linear structure on each of $\mathbb{H}^4$ and deS and in particular planes through the origin cut $\mathbb{H}^4$ and deS in geodesics and all geodesics are of this form (this is proved below).

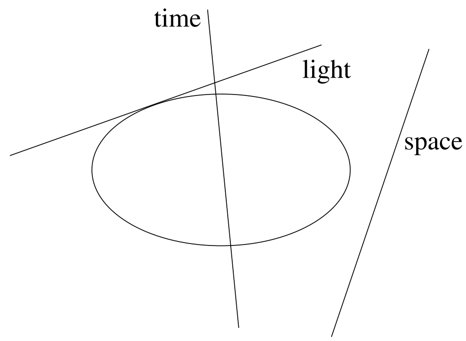

Figure 1: three types of lines

Geodesics come in three types: *light-like* corresponding to planes tangent to the light cone; *time-like* corresponding to planes which intersect $\mathbb{H}^4$ and *space-like* corresponding to planes which meet L only at the origin (and hence miss $\mathbb{H}^4$). Figure 1 is a projective picture illustrating these types.

Hyperbolic and de Sitter geometry

The subset $\mathbb{H}^4$ of $\mathbb{P}^4$ with the action of $\mathrm{SO}(1,4)$ is the *Klein model* of hyperbolic 4–space. S^3 is then the sphere at infinity and $\mathrm{SO}(1,4)$ acts by conformal transformations of S^3 and indeed is isomorphic to the group of such transformations. $\mathrm{SO}(1,4)$ acts as the group of time and space orientation preserving isometries of deS and is also known as the *de Sitter group* as a result.

Transitivity of points and geodesics

There is an element of $\mathrm{SO}(1,4)$ carrying any half-ray to any other of the same type (corresponding to the decomposition (1)). Here is an explicit way to see this which also proves transitivity on geodesics and checks the characterisation of geodesics mentioned earlier. A Lorentz transformation of M^2, namely $x \mapsto \left(\begin{smallmatrix} \cosh q & \sinh q \\ \sinh q & \cosh q \end{smallmatrix}\right)x$, which we shall call a *shear*, for suitable choice of $q \in \mathbb{R}$ acting in a vertical plane (one containing the x_0–axis) and crossed with the identity on the perpendicular 3–space, will move any point of $\mathbb{H}^4$ to the centre (intersection with the x_0–axis) and any point of deS to a point on the equator (intersection with the (x_1, x_2, x_3, x_4)–space). Then a rotation about the x_0–axis carries it to any other point. This proves transitivity for points in $\mathbb{H}^4$ (and similarly $\mathbb{H}^4_q$) and deS. For S^3 (and similarly S^3_q) a rotation about the x_0–axis carries one point to any other.

We now extend this argument to prove transitivity on geodesics. Since we have not yet proved that geodesics are intersections with planes through the origin we shall call such intersections *lines* and we shall see that lines are in fact geodesics shortly. Notice that a time-like plane (one meeting $\mathbb{H}^4$, see terminology introduced above) meets deS in *two* antipodally opposite lines which get identified in $\mathbb{P}^4$ and which we call a line-pair.

We prove transitivity for lines of the same type. In $\mathbb{H}^4$ all lines are time-like and we can move any point of each line to the centre and rotate to carry one line into the other. Turning now to de Sitter space, the same sequence of transformations applied to deS carries any time-like line-pair to any other and a rotation can be used to swap the lines if necessary. A space-like line can be carried to the equator by two perpendicular shears.

To prove transitivity for light-like lines, we observe that in Euclidean terms deS is a hyperboloid of one sheet ruled by lines and that each tangent plane to the light cone meets deS in two ruling lines. These lines are light-lines in M^5 and hence in deS. (See Figure 2, which is taken from Moschella [1].) Now we can carry a point on a given light-line to a point of the equator by a shear followed by a rotation. But the rotations fixing this point now carry the light-line around the light cone in deS and hence any two are equivalent under an isometry.

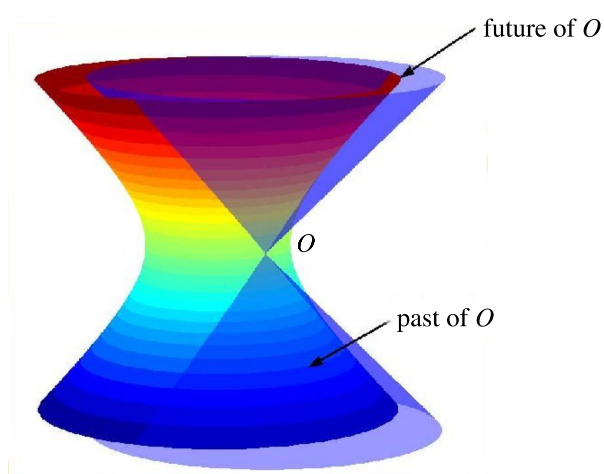

Figure 2: Light-cones: the light-cone in deS is the cone on a 0–sphere (two points) in the dimension illustrated, in fact it is the cone on a 2–sphere. The figure is reproduced from [1].

The proof of transitivity shows that there are isometries which move any line along itself and further we can now see that the isometries which fix a given line point-wise form a SO(3) subgroup of rotations. Thus by symmetry, parallel transport along a line carries a line into itself and they are, as claimed, geodesics. Since there are lines through each point in each direction, all geodesics are of this form. We have proved:

Proposition 1 *Let* l, m *be geodesics in* deS *of the same type and let* $P \in l$ *and* $Q \in m$*. Then there is an element of* $\mathrm{SO}(1,4)$ *carrying* l *to* m *and* P *to* Q*.*

The expansive metric

Let Π be the 4–dimensional hyperplane $x_0 + x_4 = 0$. This cuts deS into two identical regions. Concentrate on the upper complemetary region Exp defined by $x_0 + x_4 > 0$. Π is tangent to both spheres at infinity S^3 and S^3_q. Name the points of tangency as P on S^3 and P' on S^3_q. The hyperplanes parallel to Π, given by $x_0 + x_4 = k$ for $k > 0$, are also all tangent to S^3 and S^3_q at P, P' and foliate Exp by paraboloids. Denote this foliation by $\mathcal{F}$. We shall see that each leaf of $\mathcal{F}$ is in fact isometric to $\mathbb{R}^3$. There is a transverse foliation by the time-like geodesics passing through P and P'.

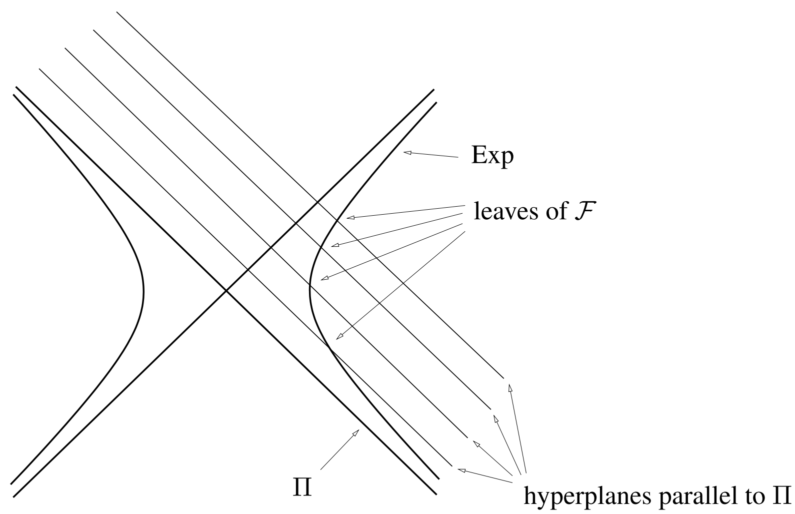

Figure 3: The foliation $\mathcal{F}$ in the (x_0, x_4)–plane

These foliations are illustrated in Figures 3 and 4. Figure 3 is the slice by the (x_0, x_4)–coordinate plane and Figure 4 (the left-hand figure) shows the view from the x_4–axis in 3–dimensional Minkowski space (2–dimensional de Sitter space). This figure and its companion are again taken from Moschella [1].

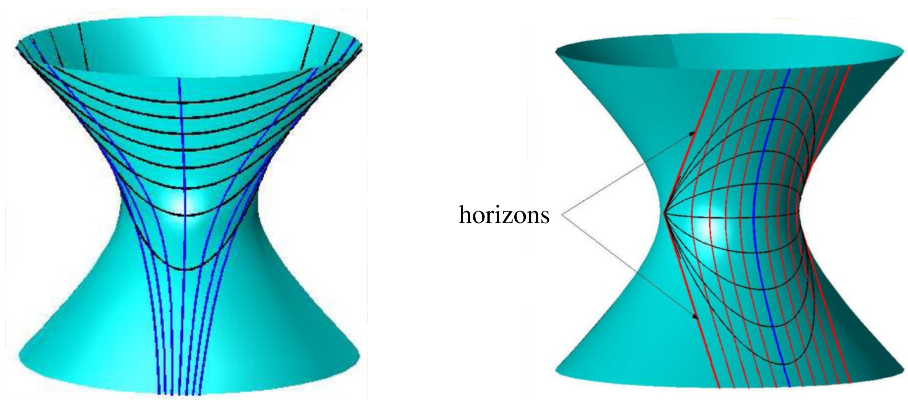

Figure 4: Two figures reproduced from [1]. The left-hand figure shows the foliation $\mathcal{F}$ (black lines) and the transverse foliation by geodesics (blue lines). The righthand figure shows the de Sitter metric as a subset of deS.

Let G be the subgroup of the Lorentz group which fixes P (and hence P' and Π). G acts on Exp. It preserves both foliations: for the second foliation this is obvious, but all Lorentz transformations are affine and hance carry parallel hyperplanes to parallel hyperplanes; this proves that it preserves the first foliation. Furthermore affine considerations also imply that it acts on the leaves of $\mathcal{F}$ by scaling from the origin: in other words there is a scale factor $\lambda(g)$ for each $g \in G$ which maps the leaf at distance μ from Π to the one at distance $\lambda(g)\mu$ (here distance means Euclidean distance in $\mathbb{R}^5$). The map $\lambda\colon\ G \to \mathbb{R}_*$ is a homomorphism from G to the positive reals under multiplication.

Now consider the action of G on S^3 (the light sphere at infinity). As remarked earlier $\mathrm{SO}(1,4)$ acts on S^3 as the group of conformal isomorphisms. Thus G is the subgroup of conformal isomorphisms of S^3 fixing the point P. But this is a very familiar group. Use stereographic projection to identify $S^3 - P$ with $\mathbb{R}^3$ then G acts as the group of conformal isomorphisms of $\mathbb{R}^3$, which is precisely the group of similarities of $\mathbb{R}^3$ ie the group generated by isometries and dilations. There is then another homomorphism $\sigma\colon\ G \to \mathbb{R}_*$ which maps each element to its scale factor. It can be checked that $\lambda = \sigma$. One way to see this is to list the non-trivial normal subgroups of G (this is not hard) and check that there is only one homomorphism to $\mathbb{R}_*$. Another way is to check directly that a standard shear j, say, in the (x_0, x_4)–plane is a pure dilation at infinity by exactly $\lambda(j)$. (It is easy to see that j acts on leaves by dilation to first order and at infinity. The scaling is given by affine considerations.)

Now define H to be the kernel of $\lambda = \sigma$. This comprises elements of G which maps the leaves of $\mathcal{F}$ into themselves, and in terms of the action on $S^3 - P = \mathbb{R}^3$, it is the subgroup of Euclidean isometries. Thus the Euclidean group acts on each leaf of $\mathcal{F}$. By dimension considerations it acts an the full group of isometries of each leaf. It follows that each leaf has a flat Euclidean metric.

We can now see that the metric on Exp is the same as the Robertson–Walker metric for a uniformly expanding infinite universe. The transverse foliation by time-like geodesics determines the standard observer field and the distance beetween hyperplanes defining $\mathcal{F}$ gives a logarithmic measure of time. Coordinates are given below. Notice that we have proved that every isometry of Exp is induced by an isometry of deS.

It is worth remarking that exactly the same analysis can be carried out for $\mathbb{H}^4$ where the leaves of the foliation given by the same set of hyperplanes are again Euclidean. This gives the usual "half-space" model for hyperbolic geometry with Euclidean horizontal sections and vertical dilation.

Time-like geodesics in Exp

We have a family of time-like geodesics built in to Exp namely the observer field mentioned above. These geodesics are all *static*. They are all equivalent by a symmetry of Exp because we can use a Euclidean motion to move any point of one leaf into any other point. Other time-like geodesics are *non-static*. Here is a perhaps surprising fact:

Proposition 2 *Let l, m be any two non-static geodesics in* Exp*. Then there is an element of $G =$* Isom(Exp) *carrying l to m.*

Thus there in no concept of conserved velocity of a geodesic with respect to the standard observer field in the expansive metric. This fact is important for the analysis of black holes in a universe with a fixed cosmological constant, cf [2].

The proof is easy if you think in terms of hyperbolic geometry. Time-like geodesics in Exp are in bijection with geodesics in $\mathbb{H}^4$ since both correspond to 2–planes through the origin which meet $\mathbb{H}^4$. But if we use the upper half-space picture for $\mathbb{H}^4$ static means vertical and two non-static geodesics are represented by semi-circles perpendicular to the boundary. Then there is a conformal map of this boundary (ie a similarity transformation) carrying any two points to any two other: translate to make one point coincide and then dilate and rotate to get the other ones to coincide.

The de Sitter metric

There is another standard metric inside de Sitter space which is the metric which de Sitter himself used. This is illustrated in Figure 4 on the right. This metric is static, in other words there is a time-like Killing vector field (one whose associated flow is an isometry). The region where it is defined is the intersection of $x_0 + x_4 > 0$ defining Exp with $x_0 - x_4 < 0$ (defining the reflection of Exp in the (x_1, x_2, x_3, x_4)–coordinate hyperplane). The observer field, given by the Killing vector field, has exactly one geodesic leaf, namely the central (blue) geodesic. The other leaves (red) are intersections with parallel planes not passing through the origin. There are two families of symmetries of this subset: an SO(3)–family of rotations about the central geodesic and shear along this geodesic (in the (x_0, x_1)–plane). Both are induced by isometries of deS.

This metric accurately describes the middle distance neighbourhood of a black hole in empty space with a nonzero cosmological constant. The embedding in deS is determined by the choice of central time-like geodesic. Proposition 2 then implies that there are precisely two types of black hole in a standard uniformly expanding universe.

Explicit formulae

Here are explicit formulae for these two metrics in terms of the x_i adapted from wikipedia.

Expansive metric Let

$$\begin{aligned} x_0 &= a\sinh(t/a) + r^2\exp(t/a)/2a \\ x_4 &= a\cosh(t/a) - r^2\exp(t/a)/2a \\ x_i &= \exp(t/a)\, y_i \quad \text{for} \quad 1 \le i \le 3 \end{aligned}$$

where $r^2 = \sum_i y_i^2$. Then in the (t, y_i) coordinates the metric reads

$$ds^2 = -dt^2 + \exp(2t/a)\, du^2$$

where $du^2 = \sum dy_i^2$ is the flat metric on y_i's.

de Sitter coordinates (static coordinates) Let

$$\begin{aligned} x_0 &= \sqrt{a^2 - r^2}\sinh(t/a) \\ x_4 &= \sqrt{a^2 - r^2}\cosh(t/a) \\ x_i &= r\, z_i \quad \text{for} \quad 1 \le i \le 3 \end{aligned}$$

where z_i gives the standard embedding of S^2 in $\mathbb{R}^3$ then in r, z_i coordinates the metric reads

$$ds^2 = -Qdt^2 + 1/Qdr^2 + r^2 d\Omega^2 \tag{2}$$

where $Q = 1 - (r/a)^2$ and $d\Omega^2$ is the standard metric on S^2.

References

[1] **U Moschella**, The de Sitter and anti-de Sitter sightseeing tour, Séminaire Poincaré 1 (2005) 1–12, available at http://www.bourbaphy.fr/moschella.pdf

[2] **R MacKay**, **C Rourke**, *Natural observer fields and redshift*, J Cosmology 15 (2011) 6079–6099

[3] http://en.wikipedia.org/wiki/De_Sitter_space