Angular momentum, inertial drag and tangential velocity in a rotating dynamical system

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A simple piece of dynamics which applies to any stationary axially symmetric system has some startling consequences for cosmology.

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For Robert MacKay on the occasion of his 60^{th} *birthday*

I want to describe a very simple piece of dynamics which is not well known and which has some startling consequences for cosmology.

The background is the well-known property of general relativity, and indeed of any sensible dynamical theory, that there is an inertial frame at every point, well-defined up to uniform linear motion. Such a frame is characterised by the lack of forces correlated with acceleration or rotation. It is a property of general relativity (and presumably of other dynamical theories) that these inertial frames are affected by the surrounding dynamics and the particular case that we are going to consider is rotation. If you have a body which is rotating, then it will drag the inertial frames for points near it so as to induce a coherent rotation (ie a rotation about a parallel axis, perhaps in the opposite sense). We refer to this frame dragging as *inertial drag*.

From now on we do not assume that we have any particular dynamical theory but merely one given by a metric (in the same way as general relativity) with these two properties (existence of inertial frames and inertial drag). We shall assume that our metric is stationary, which you can think of as "looking much the same at all times" (technically there is a killing asymptotically timelike vector field), at least where we are working.

Now suppose we have a heavy axially symmetric body (perhaps a black hole) which is rotating about its axis of symmetry. We use Euclidean coordinates to describe motion near this body without supposing that the metric is Euclidean. These coordinates provide a background metric against which to measure inertial drag and we will suppose that these coordinates describe a distant observer's view of the space-time near the body. In these coordinates the axis of symmetry is the z–axis and positive rotation is anticlockwise (ie from x to y) when viewed from above (from the positive z direction) and for simplicity assume the body has the (x, y)–plane as a plane of symmetry (think for example of a ball centred at the origin). Use coordinates (r, ϕ) in the (x, y)–plane and assume (by scaling in the r direction if necessary) that the circles $z = r =$ const all have the correct Euclidean metric (ie length $2\pi r$).

We are going to restrict attention to the behaviour of a test particle P in the (x, y), or (r, ϕ)– plane. The most general metric satisfying our constraints, near the (x, y)–plane, is a continuously variable spherically symmetric metric $\mu(r)$ defined near the sphere of radius r with a continuously variable rotation $\omega(r)$ superimposed (ie replace ϕ by $\phi - \omega(r)\,t$. This can be seen because all we need to think about is a neighbourhood of the circle of radius r in the (x, y)–plane and we have enough freedom to fit any stationary axially symmetric metric near this circle. For definiteness, you can think of $\mu(r)$ as being the Schwarzschild metric for all r and the global metric as obtained from the Schwarzschild metric by making the global substitution $\phi - \omega(r)\,t$ for ϕ. But the arguments we shall present apply to the most general metric of this type.

Let v be the tangential velocity of P (ie the velocity in the ϕ direction along the circle of radius r). We will show that there is always a simple relationship between v and r depending on $\omega(r)$. The key to this is the effect of inertial drag on angular momentum. First notice that the effect of the substitution of $\phi - \omega(r)\,t$ for ϕ is that the inertial frame at r is now rotating with angular velocity $\omega(r)$ about the origin. This is obvious if ω is constant because all frames are now rotating with angular velocity ω, and since inertial frames are local this is true locally if ω depends on r. Now inertial frames are only defined up to uniform linear motion, and hence we can choose to regard the inertial frame at r as rotating about the origin and it is helpful to think of these inertial frames as transparent sheets pinned together at the origin with each rotating at its own angular velocity. Thus each inertial frame has the same point-set but is rotating with a different angular velocity about the origin. Each sheet corresponds to a particular value of r. We need to be very clear about the nature of motion in one of these frames. We will say that a particle moving with a frame (ie one stationary in that frame) has no *inertial velocity* and call its velocity *rotational*. In general if a particle has velocity $\mathbf{v}$ (measured in the background space) then we can write

$$\mathbf{v} = \mathbf{v}_{\text{rot}} + \mathbf{v}_{\text{inert}}$$

where its *rotational velocity* $\mathbf{v}_{\text{rot}}$ is the velocity due to rotation of the local inertial frame and $\mathbf{v}_{\text{inert}}$ is its *inertial velocity* which is the same as its velocity measured *in* the local inertial frame. Note that $\mathbf{v}_{\text{rot}} = r\omega(r)$ directed along the tangent. Inertial velocity correlates with the usual Newtonian concepts of centrifugal force and conservation of angular momentum. Indeed the proof of conservation of angular momentum given by

Newton uses only that the force (in Newton's case gravitational attraction by the heavy body) acts towards the centre and that the r–coordinate gives the correct Euclidean distance along r–circles. It therefore applies to any situation where there is a central force or space-time geometry which simulates a central force and in particular to inertial motion in these inertial frames.

As a particle moves in the equatorial plane it moves between the sheets so that a rotation about the origin which is rotational in one sheet becomes partly inertial in a nearby sheet. For definiteness, suppose that $\omega(r)$ is a decreasing function of r and consider a particle moving away from the origin and at the same time rotating counter-clockwise about the origin. The particle will appear to be being rotated by the sheet that it is in and this causes a tangential acceleration. We call this acceleration the *slingshot effect* because of the analogy with the familiar effect of releasing an object swinging on a string. But at the same time the particle is moving to a sheet where the rotation due to inertial drag is decreased and hence part of the tangential velocity becomes inertial and is affected by conservation of angular momentum which tends to decrease the angular velocity. More precisely the slingshot effect produces an acceleration $dv/dr = \omega(r)$. On the other hand $v_{\text{inert}} = v - \omega r$ is the "inertial" tangential velocity (corrected for rotation of the local inertial frame) and therefore conservation of angular momentum produces a deceleration in v of v_{inert}/r or an acceleration $dv/dr = \omega - v/r$. Adding the two effects gives the fundamental relation:

$$\boxed{\frac{dv}{dr} = 2\omega - \frac{v}{r}} \tag{1}$$

There is a rigorous proof of this relation given in [4, II, page 7ff], adapting Newton's proof, and which applies to the present hypotheses.

From the fundamental relation you can derive a simple relation between v and r as follows. Given ω as a function of r, (1) can be solved to give v as a function of r. We can rewrite it as

$$r\frac{dv}{dr} + v = 2\omega r\,.$$

The LHS is $d/dr\,(rv)$ and we obtain the general solution:

$$\boxed{v = \frac{1}{r}\Big(\int 2\omega r\,dr + \text{const}\Big)} \tag{2}$$

This is the simple piece of dynamics, which, as we have seen, applies to stationary rotating dynamical systems under very general conditions. We now turn to consequences.

Consequence 1 Quasars

Suppose that our central mass is in fact a black hole which is a typical hypothesis for the nature of quasars.

From (2) we can read the angular momentum per unit mass of our particle as $vr = \int 2\omega r\, dr + \text{const}$. By suitable choice of the integration constant, we can find solutions with low angular velocity for r small and significant angular velocity for larger r and it follows that the effect of the inertial drag is that *the rotating body can absorb angular momentum*. And notice that this holds for almost any dynamical theory, in particular general relativity. This may not seem like a startling consequence until you look at the history of quasar research (or black hole research, which is much the same thing). For a general overview see Meier [2].

It is received wisdom in this subject that angular momentum presents a serious obstruction to accretion of matter by a black hole and this has led to the subject being dominated by the theory of accretion discs. This obstruction to accretion was found very early in the subject, for example Michel [3, Section 4, p 158] (1976) states:

> ... *One must, however, somehow transfer away most of the angular momentum that the infalling gas had relative to the centre of mass. It seems physically plausible that the effect of such angular momentum would be to choke down the inflow rates. For example, even when magnetic torques are included* ... *one finds that the 'infall' solutions terminate at finite distances from the origin in analogy with the minimum approach distance of a single particle trajectory having non-zero initial angular momentum.* ...

A consequence of this is that redshift, which is frequently observed in quasar radiation, is generally believed to be cosmological and not intrinsic or gravitational. Indeed, if the observed radiation comes from an accretion disc affected by local gravitational effects, there would be wide spectral lines (redshift gradient) and not narrow ones as observed.

But if angular momentum can be nullified by central rotation, then it does not force the existence of an accretion disc and redshift can be largely gravitational. Moreover there is a feedback effect working in favour of this. If the incoming matter has excess angular momentum, then it will tend to contribute to the central rotation which therefore changes to increase the inertial drag effect until the two balance again. Conversely, if there is a shortfall, the black hole will slow down. In other words, once locked on the ambient conditions that allow the black hole to accrete, there is a mechanism for maintaining that state.

The conclusion is that we can effectively ignore the angular momentum obstruction for accretion.

If we do ignore the angular momentum problem and study spherically symmetric accretion models then we find that gravitational redshift can take arbitrary values. This model is studied in joint work with Robert MacKay and Rosemberg Toala Enriques [5]. We find that redshift is given by

$$z = 1.27 \times 10^7 \, P^{-1} \, n^{-1} \, T^{1.5} \, [1/(2X)] \tag{3}$$

where P is the black hole mass in solar masses, n is density of the ambient gas/plasma in number of particles per cubic metre and T is temperature in degrees Kelvin. X is an absorption factor which can be taken to be $1/2$ (ie ingore the factor in square brackets). In [5] many worked examples are given and the theory fits with the (currently discredited) observations of Arp et al [1] which give convincing examples of intrinsic redshift.

Consequence 2 The rotation curve

For full details here, see [4, II]. Looking at (2) we can read that the rotation curve for our particle (ie the plot of v against r) has a flat asymptote iff ω is asymptotically equal to a C/r where C is constant. There is supporting evidence for this behaviour presented in [4, I].

Thus we have a model for observed rotation curves without assuming any problematic “dark matter”.

Consequence 3 The quasar galaxy spectrum

Carrying on the story from Consequence 1, a property of the model given in [5] is that radiation comes principally from the Eddington sphere and is subject to a large reduction in energy (a factor $(1+z)^{-2}$) if redshift is big (which it typically is for small black holes: for example observations of Sgr A* are fitted with a redshift of $z = 10^4$!). This implies that virtually all the infalling energy is absorbed by the central black hole. Thus the ability of a rotating black hole to cancel out angular momentum allows it to “feed” on the surrounding medium and grow. (3) implies that, as it grows, redshift decreases (as directly observed by Arp) and the proportion of energy reaching the centre decreases. The surplus accumulates near the event horizon and chokes the inflow as Michel suggests and a toroidal accretion structure (which we shall call the *belt*), similar to the conventional theory, forms. The quasar starts to produce explosive outflow (jets) and to morph into an “active galaxy”. The angular momentum locking effect is now no longer stable and another stable configuration forms.

Consequence 4 Spiral structure

This new stable configuration happens because the jets carry away angular momentum and cause the whole system to rotate in the opposite direction to the rotation of the belt. So we have the central black hole rotating to the left (say) and a surrounding belt rotating to the right. This implies that the angular momentum in the inertial frames is augmented by the inertial drag effects described earlier and the effective energy in the belt similarly augmented. Thus energy is being fed into the belt structure directly from the black hole itself. The jets now become massive and permanently established and manifest themselves as the familiar spiral arms of the galaxy. Furthermore the inertial drag effect, which causes the flat rotation curve, causes angular momentum to be lost in the other direction and a balance is reached. This balance explains why rotation velocity is roughly constant over all galaxies as observed. For more detail and for accurate models see [4]. There is no further growth in general, indeed there is now steady loss due to radiation energy and matter lost to the system, to balance accretion.

So we have a smooth transition from small black hole (quasar) through active and on to full-size spiral galaxies. Combining this with the observations also due to Arp of active galaxies apparently giving birth to quasars, we have evidence of a life-form, perhaps the dominant life-form of the universe.

Here is a rough guide to the central masses for the various stages of this spectrum: quasars, 10^6–10^8 solar masses; active galaxies 10^8–10^{10} solar masses; full size spirals, 10^{10}–10^{12} solar masses.

References

[1] **H Arp**, *Sundry articles*, available at http://www.haltonarp.com/articles

[2] **D M Meier**, *Black hole astrophysics: the engine paradigm*, Springer (2009)

[3] **F C Michel**, *Accretion of matter by condensed objects*, Astrophys and space sci 15 (1972) 152–160

[4] **C Rourke**, *Sciama's principle and the dynamics of galaxies, I–V*, available at http://msp.warwick.ac.uk/~cpr/paradigm/

[5] **C Rourke**, **R Toala Enriques**, **R S MacKay**, *Black holes, redshift and quasars*, preprint, available at http://msp.warwick.ac.uk/~cpr/paradigm/quasars.pdf

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