Uniqueness of spherically-symmetric vacuum solutions to Einstein’s equations

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We prove that spherically-symmetric vacuum solutions of Einstein’s equations with a cosmological constant ($\text{Ric} = \Lambda g$, where $\Lambda$ is not a priori assumed to be constant or independent of $t$) are necessarily static, have $\Lambda$ constant, are unique and coincide with the Schwarzschild–de Sitter metric. Relaxing the conditions on $\text{Ric}$ we find that assuming that the first off-diagonal term is zero and that the first two diagonal terms are in the correct proportion (again with $\Lambda$ not assumed constant) corresponds precisely to the static and natural observer field case discussed in [1].

Note Einstein’s equations with cosmological constant for a vacuum are usually written $\text{Ric} + (\kappa - \frac{1}{2} S) g = 0$ where $\text{Ric}$ is Ricci curvature $\kappa$ is the cosmological constant and $S$ is the scalar curvature. But this is equivalent to $\text{Ric} = \Lambda g$ where $\Lambda = \frac{1}{2} S - \kappa$ is a scalar field, which cannot a priori be assumed constant or independent of $t$.

Consider the general spherically-symmetric metric

$$ds^2 = -Q dt^2 + P dr^2 + r^2 d\Omega^2$$

where $P$ and $Q$ are positive functions of $r$ and $t$ on a suitable domain. Here $t$ is time, $r$ is radius and $d\Omega^2$, the standard metric on the 2–sphere, is an abbreviation for $d\theta^2 + \sin^2 \theta d\phi^2$ (or more symmetrically for $\sum_{j=1}^{3} dz_j^2$ restricted to $\sum_{j=1}^{3} z_j^2 = 1$). The Schwarzschild–de Sitter metric is the case $Q = 1/P = 1 - \Lambda r^2/3 - 2M/r$ with $\Lambda$ and $M$ constants.

We compute the Ricci curvature using the useful primitive formulae given in [2]. We start by computing $R_{tt}, R_{tr}$ and $R_{rr}$ where $R = \text{Ric}$. From [2, eq (1) p2] we get (after a couple of lines of computation)

$$4R_{tt} = \frac{4\dot{P}}{rP}$$

and from [2, eq (2) p3]

$$\frac{4R_{tt}}{-Q} = -\frac{\dot{P}}{W^2} \frac{\dot{W}}{W} + 2\frac{\dot{P}'}{W} - \frac{4Q'}{rW} - \frac{2Q''}{W} + \frac{Q'W'}{W^2}$$
and

\[
\frac{4R_{rr}}{P} = \frac{\dot{P}}{W^2} \ddot{W} - \frac{2\dot{P}}{W} \frac{2P' - 2P''}{rP^2} - \frac{Q''}{W} + \frac{Q'W'}{W^2}.
\]

Here we have used dot for diff wrt \( t \) and prime for diff wrt \( r \) and \( W \) is short for \( PQ \).

Now assume that \( \text{Ric}_{rr} \) is zero then from (2) we get \( \dot{P} = 0 \) and \( P \) depends only on \( r \) and further the first two terms in (3) and (4) are now zero and these equations reduce to

\[
\frac{4R_{rr}}{-Q} = \frac{-4Q'}{rW} - \frac{2Q''}{W} + \frac{Q'W'}{W^2}
\]

and

\[
\frac{4R_{rr}}{P} = \frac{4P'}{rP^2} - \frac{2Q''}{W} + \frac{Q'W'}{W^2}.
\]

Note that only the first terms differ. Now suppose that \( \text{Ric}_{ii} = \Lambda g_{ii} \) for \( i = r, t \) where \( \Lambda \) is not assumed to be constant. Then the left-hand sides of these equations are the same, and hence

\[
\frac{Q'}{W} + \frac{P'}{P^2} = \frac{Q'P + P'Q}{PW} = \frac{W'}{PW} = 0,
\]

which is true iff \( W' = 0 \) ie \( W \) depends only on \( t \). If this is the case then \( Q(t, r) = W(t)/P(r) \) and we can assume that \( W = 1 \) by reparametrising \( t \) and hence \( P = 1/Q \) are functions of \( r \) alone and we are in the static case with a natural observer field discussed in [1].

In this case, \( \text{Ric} = \Lambda g \) implies that \( \Lambda \) is also a function of \( r \) alone and from either equation we read

\[
-2\frac{Q'}{r} - \frac{Q''}{W} = 2\Lambda.
\]

This DE is easily solved. Since \( (r^2Q')' = 2rQ' + r^2Q'' \) it is equivalent to

\[
r^2Q' = -\int 2r^2\Lambda
\]

and we have an explicit formula for \( Q \). (In the case \( \Lambda \) is constant we immediately get the Schwarzschild–de Sitter metric up to careful choice of integration constants.)

Now assume that \( \text{Ric} = \Lambda g \) for all components. Inspecting [2, eq (1) p2] it is easy to see that all off-diagonal entries are zero (indeed in the general case, only \( R_{rt} = R_{tr} \) can be nonzero and we have seen that this is zero in the static case). So the only new content is that \( R_{ii} = \Lambda g_{ii} \) for \( i = \phi, \theta \). By spherical symmetry the two coordinates give exactly the same information and we examine just the case \( i = \phi \).

We use [2, eq (2) p3] again with \( \mu = \phi \). Since no other coefficient of \( g_{ii} \) involves \( \phi \) the equation reduces to the second term in the summation and further looking at the last
term in the equation we see that we must have $\sigma = r$ or $\theta$. We quickly compute the $r$ term to be $-4 \sin^2(\theta)(Q + rQ')$ and the $\theta$ term reduces to $4 \sin^2(\theta)$. Thus $R_{\phi\phi} = \Lambda g_{\phi\phi}$ iff

\begin{equation}
1 - Q - rQ' = r^2 \Lambda.
\end{equation}

Since $(rQ)' = Q + rQ'$ this implies

\begin{equation}
rQ = r - \int r^2 \Lambda.
\end{equation}

We can eliminate $\int r^2 \Lambda$ between (8) and (10) to find

\[ r^2 Q' - 2rQ = -2r + C \]

where $C$ is a constant of integration. This is readily solved to yield

\[ Q = 1 -Kr^2 - \frac{C}{3r} \]

where $K$ is another constant of integration. Finally substituting for $Q$ in (10) we find that $\Lambda = 3K$ is constant after all and we have the Schwarzschild–de Sitter metric with $3C = 2M$ and the proof is complete.

**Note** We only need the solution to exist in any open set for the argument above to work. In particular there are no boundary conditions (at a BH or at infinity) assumed.

**References**
